A Note on Convergence of Stochastic Processes

Marco Lippi*
Dipartimento di Scienze Economiche, Università di Roma "La Sapienza"

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Abstract. This note, a companion paper to Lippi (2003), contains examples showing that convergence of a sequence of stochastic processes $\xi_n(t)$ to the stochastic process $\xi(t)$ does not imply convergence of the innovation of $\xi_n(t)$ to the innovation of $\xi(t)$, irrespectively of whether $\xi_n(t) - \xi(t)$ is stationary or not, or whether $\xi_n(t)$ converges to $\xi(t)$ uniformly or not.

* Address: Dipartimento di Scienze Economiche, Via Cesalpino 12, I-00161 Roma, Italy. E-mail: marco.lippi@uniroma1.it

1. Firstly, define a subdivision of $[-\pi\ \pi]$ as an n-tuple $\boldsymbol{\lambda} = (\ \lambda_0 \quad \lambda_1 \quad \cdots \quad \lambda_n\ )$, with $-\pi = \lambda_0 < \lambda_1 < \cdots < \lambda_{n-1} < \lambda_n = \pi$. Let $\delta_{\boldsymbol{\lambda}} = \max_h[\lambda_h - \lambda_{h-1}]$. With $\boldsymbol{\lambda}$ is associated the stochastic integral sum

$$S_{\boldsymbol{\lambda},t} = \sum_{h=1}^{n} e^{it\lambda_{h-1}}[\zeta(\lambda_h) - \zeta(\lambda_{h-1})],$$

where ζ is the spectral measure associated with $\xi(t)$.

Secondly, let $\boldsymbol{\Lambda} = \{\boldsymbol{\lambda}_n,\ n \in \mathbb{N}\}$, with $\boldsymbol{\lambda}_n = (\ \lambda_{0,n} \quad \lambda_{1,n} \quad \cdots \quad \lambda_{n,n}\ )$, be a sequence of subdivisions. We assume that $\delta_{\boldsymbol{\lambda}_n} \to 0$ as $n \to \infty$.

The Spectral Representation Theorem, in the discrete-time case, states that for any sequence of subdivisions $\boldsymbol{\Lambda}$

$$\xi(t) = \lim_{n\to\infty} S(\boldsymbol{\lambda}_n, t), \tag{1}$$

in quadratic mean.

Assuming that $\xi(t)$ is regular, a process with a positive-variance innovation is approximated by a sequence of zero-innovation processes. On the other hand, see Lippi (2003), if $\xi(t)$ is regular, then convergence in (1) is not uniform. Thus one may be tempted by the idea that such lack of uniformity is an explanation of the puzzle.

However, as I show below, uniform process convergence is neither sufficient nor necessary to ensure convergence of the innovation. Nor does the fact that $S(\boldsymbol{\lambda}_n, t)$ is deterministic play any special role in the puzzle.

2. Uniform convergence is not sufficient.
Define $\xi(t)$ and $\xi_n(t)$ as follows. Let $u(t)$ be a unit-variance white noise and define $\xi(t) = u(t)$, so that $u(t)$ is the innovation of $\xi(t)$. For $n \in \mathbb{N}$, define $\alpha_n = \exp(-An)$, with $A > 0$, and $a_n : [-\pi\ \pi] \to \mathbb{R}$ as

$$a_n(\lambda) = \begin{cases} 1 \text{ if } |\lambda| \le \pi - 1/n \\ \alpha_n \text{ otherwise.} \end{cases}$$

Let

$$a_n(\lambda) = \sum_{s=-\infty}^{\infty} b_{sn} \exp(-is\lambda)$$

be the Fourier expansion of a_n and define

$$\xi_n(t) = \sum_{s=-\infty}^{\infty} b_{sn} u(t-s).$$

Obviously $\xi(t) - \xi_n(t)$ is stationary and

$$\text{var}(\xi(t) - \xi_n(t)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |1 - a_n(\lambda)|^2 d\lambda = \frac{1}{\pi}\int_{\pi-1/n}^{\pi} (1-\alpha_n)^2 d\lambda = \frac{1}{n\pi}(1-\alpha_n)^2.$$

Thus $\xi(t)-\xi_n(t)\to 0$ in q.m. as $n\to\infty$, uniformly in t. Lastly, the spectral density of $\xi_n(t)$ is $\frac{1}{2\pi}a_n^2(\lambda)$, which is positive everywhere, so that $\xi_n(t)$ is regular and, by Kolmogorov's formula (see e.g. Brockwell and Davis, 1991, p.191),

$$\begin{aligned}\mathrm{var}(u_n(t)) &= (2\pi)\exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left(\frac{1}{2\pi}a_n^2(\lambda)\right)d\lambda\right]\\ &= (2\pi)\left\{\exp\left[\frac{1}{2\pi}\log\left(\frac{1}{2\pi}\right)\left(2\pi-\frac{2}{n}\right)\right]\times\exp\left[\frac{1}{2\pi}\left(\log\left(\frac{1}{2\pi}\right)-2An\right)\frac{2}{n}\right]\right\}.\end{aligned}$$

As $n\to\infty$, $\mathrm{var}(u_n(t))\to\exp\left(-\frac{2A}{\pi}\right)<1$, so that $u_n(t)$ cannot converge to $u(t)$.

Thus, although $\xi_n(t)$ is linear, regular and jointly stationary with $\xi(t)$, convergence of $\xi_n(t)$ to $\xi(t)$, uniform of course, does not imply convergence of $u_n(t)$ to $u(t)$.

3. Uniform convergence is not necessary.

This is an example in which $\xi_n(t)$ converges to $\xi(t)$ but not uniformly. Nonetheless the innovation converges (not uniformly of course). Let $(u(t)\ v(t))'$ be a vector orthonormal white noise. Define $\xi(t)=u(t)$ and

$$\xi_n(t)=\begin{cases} v(t) \text{ if } |t|>n\\ u(t) \text{ if } |t|\le n\end{cases}$$

Both $\xi(t)$ and $\xi_n(t)$ are stationary and white noise, so that they coincide with their own innovations. Moreover, though not uniformly, $\xi_n(t)$ converges to $\xi(t)$.

Reference.

Lippi, M. (2003) Issues Concerning the Approximation Underlying the Spectral Approximation Theorem, Mimeo. Dipartimento di Scienze Economiche. Università "La Sapienza", Roma.