1.2 Stationary processes

In the previous section I have given for granted the definition of a stochastic variable. Now I want to recall it, together with other definitions. Given a probability space $(\Omega, \mathcal{F}, P)$, a *stochastic variable* on $(\Omega, \mathcal{F}, P)$ is a function $x : \Omega \to \mathbb{R}$ such that, if H is any member of the Borel σ-field on $\mathbb{R}$, then

$$\{\omega \in \Omega,\ x(\omega) \in H\} \in \mathcal{F}.$$

The stochastic variable x induces on the Borel σ-field the measure

$$\mu_x(H) = P[\{\omega,\ x(\omega) \in H\}].$$

The measure μ_x is called the *distribution* of x. The *distribution function* of x is the function $F_x : \mathbb{R} \to \mathbb{R}$ defined as

$$F_x(r) = \mu(\{\omega,\ x(\omega) \le r\}).$$

Generalization to n-dimensional stochastic vectors, their distribution as a measure on the Borel σ-field on $\mathbb{R}^n$, and their distribution function, which is defined on $\mathbb{R}^n$, are obvious. The probability measures $\mu_{t_1,\ldots,t_n}$, used in Section 1.1, are of course distributions of stochastic vectors.

For any positive integer h, the h-th moment of the stochastic variable x on $(\Omega, \mathcal{F}, P)$, is defined as

$$E(x^h) = \int_\Omega x(\omega)^h dP(\omega) = \int_{\mathbb{R}} r^h d\mu_x(r).$$

It can also be referred to as the h-th moment of the distribution μ_x.

Let us begin with the definition of a stationary process.

Definition 1.2 The process $x = \{x_t,\ t \in \mathbb{Z}\}$, defined on the probability space $(\Omega, \mathcal{F}, P)$, is *strongly stationary* if

$$P[(x_{t_1},\ x_{t_2}, \ldots,\ x_{t_n}) \in H] = P[(x_{t_1+k},\ x_{t_2+k}, \ldots,\ x_{t_n+k}) \in H],$$

for any integers $t_1 < t_2 < \cdots < t_n$ and k, where H is any measurable subset of $\mathbb{R}^n$, that is if

$$\mu_{t_1,t_2,\ldots,t_n} = \mu_{t_1+k,t_2+k,\ldots,t_n+k}.$$

Definition 1.2 obviously implies that μ_t does not depend on t. It does not imply that μ_{t_1,t_2} does not depend on t_1 and t_2. What it says is that if we translate the couple $(t_1,\ t_2)$ by any integer k the distribution does not change, i.e. that μ_{t_1,t_2} depend only on $t_1 - t_2$.

Another obvious implication is that if, for a particular τ, the variable x_τ has finite first and second moment then x_t has finite first and second moment for all $t \in \mathbb{Z}$, and that such moments are independent of t.

Exercise 1.3 Prove that if the stochastic variable x has finite second moment, i.e. if

$$\int_\Omega x(\omega)^2 dP(\omega) < \infty,$$

then it has finite first moment. Hint: Note that $\Omega = \{\omega,\ x(\omega) \leq 1\} \cup \{\omega,\ x(\omega) > 1\}$ and split the integral of $|x|$.

Exercise 1.4 Let $\Omega = [1\ \infty)$ and assume that the distribution has density $g(r) = ar^{-\alpha}$. Of course α must be positive. But this is not sufficient to ensure that $\int_\Omega r^{-\alpha} dr$ is finite. Give the condition on α in order that the m-th moment of the distribution be finite.

Let us review some processes and see whether they are strongly stationary or not.

Example 1.5 Let x_t be strongly stationary with finite second moment, and let $y_t = a + bt + x_t$. Obviously the first moment of y_t depend on t, so that y_t is not stationary. In the same way

$$z_t = \begin{cases} \alpha + x_t \text{ if } t \leq t_0 \\ \beta + x_t \text{ if } t > t_0, \end{cases}$$

with $\alpha \neq \beta$ has not constant first moment, while $w_t = t(x_t - E(x_t))$ has constant first moment but varying second moment.

Example 1.6 Consider the process x_t defined in Example 1.3. The first moment is

$$E(x_t) = E(a)\cos\phi t + E(b)\sin\phi t,$$

which is not constant unless $E(a) = E(b) = 0$, or $\phi = 0$. The second moment is

$$E(x_t^2) = E(a^2)\cos^2\phi t + E(b^2)\sin^2\phi t + E(ab)\cos\phi t\sin\phi t.$$

Assuming that

$$E(a) = E(b) = 0, \quad E(a^2) = E(b^2), \quad E(ab) = 0, \tag{1.7}$$

first and second moment are independent of t. Moreover, as is easily proved,

$$E(x_t x_{t-k}) = E(a^2)\cos\phi k,$$

and is therefore independent of t. This is sufficient for weak stationarity (see Definition 1.3 below) but not for strong stationarity. For example, taking $\phi = \pi$ and assuming that a has finite third moment, the third moment of x_t is $E(a^3)(-1)^t$, which is time-varying unless $E(a^3) = 0$. In general, even enhancing (1.7) by assuming independence of a and b, the h-th moment, $h > 2$, of x_t can be time-varying (check).

Example 1.7 In the process defined in Example 1.4 assume conditions (1.7) and that ϕ is independent of a and b. Note that in this case

$$E(x_t x_{t-k}) = E(a^2)E(\cos\phi k).$$

Note that if ϕ is uniformly distributed in $[-\pi\ \pi]$ then

$$E(x_t x_{t-k}) = 0$$

for $k \neq 0$. Considerations analogous to those of Example 1.6, regarding strong stationarity, apply here.

Example 1.8 The tossing-coin process defined in Example 1.1 is obviously strongly stationary. Finite moving averages of the tossing-coin process are strongly stationary. The infinite moving average (1.6) is strongly stationary.

Example 1.9 Finite moving averages of a strongly stationary process are strongly stationary.

Irrespectively of whether the process x is stationary or not, if the second moment of x_{t_1} and x_{t_2} are finite then the expectation of the product $x_{t_1}x_{t_2}$ is finite. For,

$$\left|\int_\Omega x_{t_1}(\omega)x_{t_2}(\omega)dP(\omega)\right| \leq \sqrt{\int_\Omega x_{t_1}(\omega)^2 dP(\omega)}\sqrt{\int_\Omega x_{t_2}(\omega)^2 dP(\omega)},$$

or,

$$\begin{aligned}\left|\int_{\mathbb{R}} r_1 r_2 d\mu_{t_1,t_2}(r_1,r_2)\right| &\leq \sqrt{\int_{\mathbb{R}} r_1^2 d\mu_{t_1,t_2}(r_1,t_2)}\sqrt{\int_{\mathbb{R}} r_2^2 d\mu_{t_1,t_2}(r_1,t_2)} \\ &= \sqrt{\int_{\mathbb{R}} r^2 d\mu_{t_1}(r)}\sqrt{\int_{\mathbb{R}} r^2 d\mu_{t_2}(r)}\end{aligned}$$

(this is one of the forms taken by the Cauchy-Schwartz inequality; see Section 1.3.9 below). If the second moment of x_t is finite for all t, then obviously the mean $E(x_t)$, the variance $\mathrm{var}(x_t) = E[(x_t - E(x_t)^2] = E(x_t^2) - E(x_t)^2$ and the covariance $\mathrm{cov}(x_{t_1}, x_{t_2}) = E[(x_{t_1} - E(x_{t_1}))(x_{t_2} - E(x_{t_2}))]$ are finite for all t, t_1 and t_2.

Definition 1.3 The process $\{x_t,\ t \in \mathbb{Z}\}$ is *weakly stationary* if (1) the second moment of x_t is finite for all t, (2) the first moment of x_t is independent of t, (3) the cross moment $E(x_{t_1} x_{t_2})$ depends only on $t_1 - t_2$, this implying that the second moment of x_t is independent of t.

Given a weakly stationary process x_t, the function $\gamma : \mathbb{Z} \to \mathbb{R}$, defined as $\gamma_k = \mathrm{cov}(x_t, x_{t-k})$ is called *autocovariance function*. Since $\gamma_k = \gamma_{-k}$ (an easy consequence of weak stationarity), the autocovariance function is usually plotted only for $k \geq 0$. Note that strong stationarity implies weak stationarity, but only under the assumption that the second moment of x_t is finite for all t.

Observation 1.1 Obviously constancy of second moments does not imply constant distributions, so that weak stationarity does not imply strong stationarity. See Examples 1.6 and 1.7.

These Lecture Notes, as well as many books on stationary processes, concentrate on weak stationarity. This does not mean that we are particularly interested in processes that are weakly but not strongly stationary. Rather, we are interested in those properties of stationary processes that depend only on their first and second moments (provided that they are finite of course).

The variables belonging to a weakly stationary process are members of the space $L^2(\Omega, \mathcal{F}, P)$, the space of square integrable functions. Analysis of this and other infinite-dimensional spaces will be the subject of a fairly long mathematical section.

Summary. The well-known definitions of strongly and weakly stationary processes have been recalled. Strong implies weak stationarity if the second moments are finite. These Lectures concentrate on weakly stationary processes, that is on those properties of stationary processes, with finite second moments, that depend only on second moments.