Non-stationary processes

Examples:

1. Let x_t be a zero-mean stationary process and

$$y_t = a + bt + x_t$$

This process is called Trend Stationary, TS.

2. Let x_t be a zero-mean stationary process and y_t be a solution of

$$y_t = b + y_{t-1} + x_t$$

This process is called Difference Stationary, DS. An additional condition for the definition of DS processes will be specified below.

With a TS process we run a regression of y_t on 1 and t and analyze the residual using the theory of stationary processes.
With a DS process we apply the theory of stationary processes to

$$y_t - y_{t-1} = (1 - L)y_t = b + x_t$$

Non-stationary processes

Remember that non-stationarity does not mean necessarily that the process has a trend. This process

$$y_t = \begin{cases} x_t \text{ if } t \leq 0 \\ 1 + x_t \text{ if } t > 0 \end{cases}$$

represents a regime-change (the mean suddenly changes at $t = 0$).

Non-stationary processes

Analysis of the TS process is elementary. Here the source of non stationarity is a deterministic function of time which has no relationship with the stationary component. Trend and cycle do not interact.

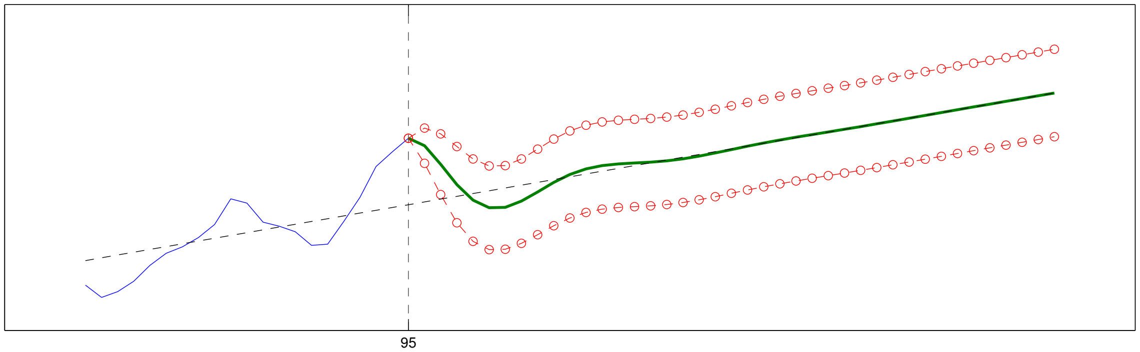
The prediction of the TS process is

$$y_{t+h|t} = a + b(t+h) + x_{t+h|t}$$

Since $x_{t+h|t} \longrightarrow 0$ as $h \longrightarrow \infty$, the long-run prediction of y_{t+h} is

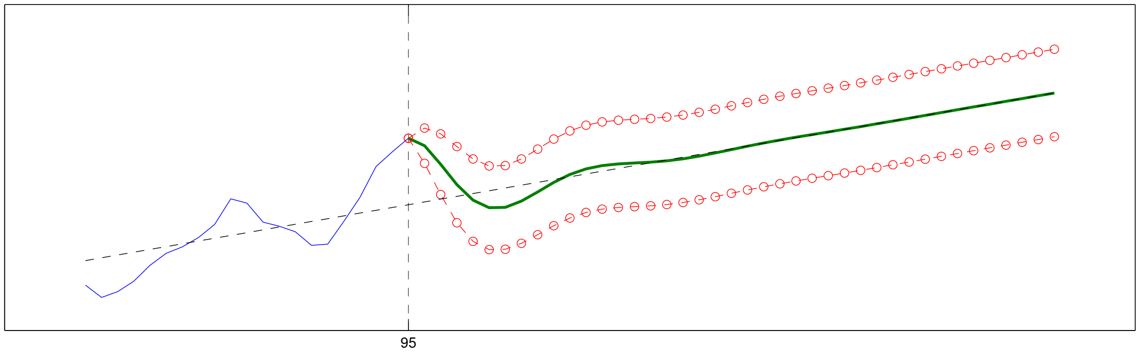
$$a + b(t+h)$$

Non-stationary processes



Prediction of the TS process with x_t driven by $(1 - 1.4L + 0.66L^2)z_t = u_t$. The blue line has some of the values of y_t, up to $t = 95$. The green line has 40 predicted values. You see how predicted values approach the trend as h increases.

Non-stationary processes



We can write

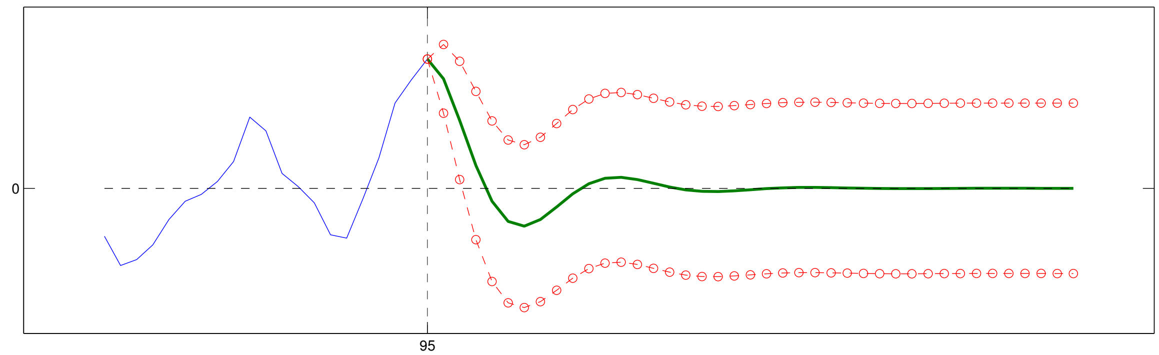
$$x_t = u_t + b_1 u_{t-1} + b_2 u_{t-2} + \cdots$$

where $1 + b_1 L + b_2 L^2 + \cdots = (1 - 1.4L + 0.66L^2)^{-1}$. The prediction error h step ahead is

$$u_{t+h} + \cdots + b_h u_{t+1}$$

whose variance is $\sigma_u^2(1 + b_1^2 + \cdots + b_h^2)$. The red lines in the figures have the predicted value plus and minus two standard deviations of the prediction error, which, if the process is normal, represents a 95% confidence interval.

Non-stationary processes



If we remove the trend $a + bt$ we find basically the same picture. This is the prediction of x_t (the scale on the vertical axis has changed).

Non-stationary processes

The DS case is much more complicated. Here the source of non-stationarity is not a deterministic component.

$$
\begin{aligned}
y_1 &= y_0 + b + x_1 \\
y_2 &= y_0 + b2 + x_1 + x_2 \\
&\vdots \\
y_t &= y_0 + bt + x_1 + x_2 + \cdots + x_t
\end{aligned}
$$

that is

$$y_t = y_0 + bt + (1 + L + \cdots + L_{t-1})x_t = y_0 + bt + \frac{1 - L^t}{1 - L} x_t$$

If

$$x_t = u_t + c_1 u_{t-1} + \cdots = c(L) u_t$$

then

$$y_t = y_0 + bt + \frac{1 - L^t}{1 - L} c(L) u_t$$

Non-stationary processes

Rewrite

$$y_t = y_0 + bt + \frac{1 - L^t}{1 - L} c(L) u_t$$

Now suppose that $c(L) = (1 - L)d(L)$, which implies that $c(1) = 0$. We have

$$y_t = y_0 + bt + \frac{1 - L^t}{1 - L} c(L) u_t = y_0 + bt + (1 - L^t) d(L) u_t = (y_0 - d(L) u_0) + bt + d(L) u_t = a + bt + d(L) u_t$$

with $d(L)u_t$ stationary. But then y_t is TS.

So let us redefine a DS process as

$$y_t = b + y_{t-1} + c(L) u_t = b + y_{t-1} + (1 + c_1 L + c_2 L^2 + \cdots) u_t$$

where

$$c(1) = 1 + c_1 + c_2 + \cdots \neq 0$$

Non-stationary processes

Example: $c(L) = 1$. This is

$$y_t = y_{t-1} + b + u_t$$

and is called random walk with drift.

$$y_t = y_0 + bt + u_1 + u_2 + \cdots + u_t$$

Usually we assume that the process starts at $t = 1$ and that y_0 is a given non-stochastic value. Then, conditional on y_0, the mean of y_t is $y_0 + bt$ and the variance is

$$t\sigma_u^2$$

The deterministic trend $y_0 + bt$ has little importance: The variance around the trend explodes (unlike the TS case).

Non-stationary processes

It is important to appreciate the difference with the stationary case:

$$y_t = b + \alpha y_{t-1} + u_t$$

with $|\alpha| < 1$ Assume the same approach, that is suppose that the process starts at $t = 1$ (in the second line you have $\alpha = 1$ for comparison:

$$\begin{aligned} y_t &= [\alpha^t y_0 + b(1 + \alpha + \cdots + \alpha^{t-1}] + [\alpha^{t-1} u_1 + \cdots + \alpha u_{t-1} + u_t] \\ (y_t &= \quad y_0 + b(1 + 1 + \cdots + \ 1 \quad) + \quad u_1 + \cdots + \ u_{t-1} + u_t) \end{aligned}$$

As $t \longrightarrow \infty$ this converges to the stationary solution

$$\frac{b}{1-\alpha} + (1 - \alpha L)^{-1} u_t$$

Note that the effect of y_0 converges to zero.

Non-stationary processes

Back to the random walk with drift

$$y_t = b + y_{t-1} + u_t$$

We have

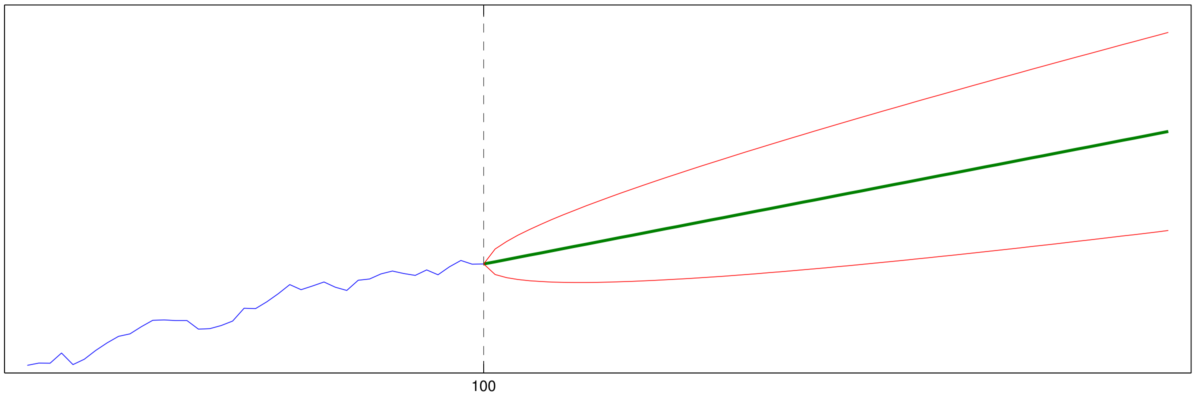
$$y_{t+h} = y_t + bh + u_{t+1} + \cdots + u_{t+h}$$

so that

$$y_{t+h|t} = y_t + bh, \qquad \mathrm{E}(y_{t+h} - y_{t+h|t})^2 = \sigma_u^2 h$$

In the TS case the influence of y_t on the predicted values tends to zero as h increases: The predicted values approach the trend $a + bt$. In the random walk the effect of y_t on the predicted values never vanishes. Moreover, the prediction error variance tends to infinity at speed h.

Non-stationary processes



In the figure:

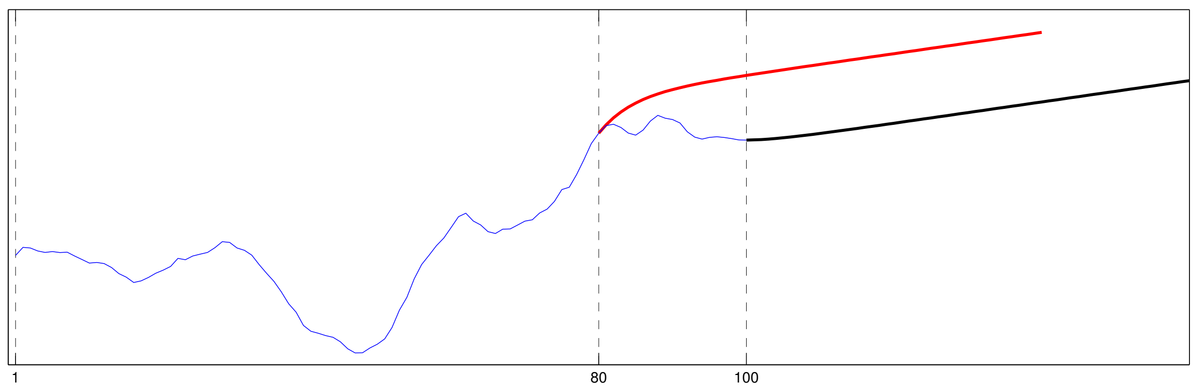
A random walk between $t = 60$ and $t = 100$, blue line.

60 predicted values, green line.

95% confidence intervals, red lines.

You see the size of the confidence interval increasing, unlike the TS case, with $\sqrt{h}$.

Non-stationary processes



In the figure

$$y_t = b + y_{t-1} + (1 - 0.8L)^{-1} u_t$$

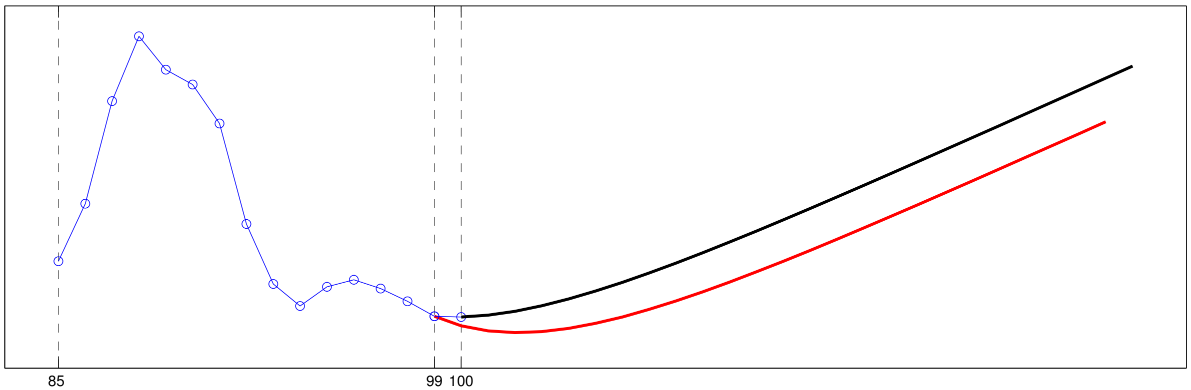
blue line.

Predicted values at $t = 80$, red line.

Predicted values at $t = 100$, black line.

You see that the predicted values tend to a trend line with the same slope, which is b, but with an intercept depending on the value of y at t, the origin of the prediction.

Non-stationary processes



Same as the previous figure, only that here the predictions are based at $t = 99$ and $t = 100$. This deserves some analysis.

$$y_{t+h} = bh + y_t + x_{t+1} + \cdots + x_{t+h}$$

Assume that

$$x_t = u_t + c_1 u_{t+1} + c_2 u_{t+2} + \cdots$$

Non-stationary processes

Rewrite

$$y_{t+h} = bh + y_t + x_{t+1} + \cdots + x_{t+h}$$

with

$$x_t = u_t + c_1 u_{t+1} + c_2 u_{t+2} + \cdots$$

Then

$$\begin{aligned} y_{t+h|t} &= bh + y_t + x_{t+1|t} + \cdots + x_{t+h|t} \\ &= bh + y_t + [c_1 u_t + c_2 u_{t-1} + \cdots] + \cdots + [c_h u_t + c_{h+1} u_{t-1} \cdots] \\ &= bh + y_t + (c_1 + c_2 + \cdots + c_h) u_t + (c_2 + c_3 + \cdots + c_{h+1}) u_{t-1} + \cdots \end{aligned}$$

Obviously

$$\begin{aligned} y_{t+h|t-1} &= b(h+1) + y_{t-1} + x_{t|t-1} + \cdots + x_{t+h|t-1} \\ &= b(h+1) + y_{t-1} + [c_1 u_{t-1} + c_2 u_{t-2} + \cdots] + \cdots + [c_h u_{t-1} + c_{h+1} u_{t-2} \cdots] \\ &= b(h+1) + y_{t-1} + (c_1 + c_2 + \cdots + c_{h+1}) u_{t-1} + (c_2 + c_3 + \cdots + c_{h+2}) u_{t-2} + \cdots \end{aligned}$$

Non-stationary processes

Again

$$
\begin{aligned}
y_{t+h|t} &= bh + y_t + x_{t+1|t} + \cdots + x_{t+h|t} \\
&= bh + y_t + (c_1 + c_2 + \cdots + c_h)u_t + (c_2 + c_3 + \cdots + c_{h+1})u_{t-1} + \cdots
\end{aligned}
$$

$$
\begin{aligned}
y_{t+h|t-1} &= b(h+1) + y_{t-1} + x_{t|t-1} + \cdots + x_{t+h|t-1} \\
&= b(h+1) + y_{t-1} + (c_1 + c_2 + \cdots + c_{h+1})u_{t-1} + (c_2 + c_3 + \cdots + c_{h+2})u_{t-2} + \cdots
\end{aligned}
$$

Then

$$
\begin{aligned}
y_{t+h|t} - y_{t+h|t-1} &= y_t - y_{t-1} - b + (c_1 + c_2 + \cdots + c_h)u_t - c_1 u_{t-1} - c_2 u_{t-2} - \cdots \\
&= x_t + (c_1 + c_2 + \cdots + c_h)u_t - c_1 u_{t-1} - c_2 u_{t-2} - \cdots \\
&= (1 + c_1 + c_2 + \cdots + c_h)u_t
\end{aligned}
$$

Non-stationary processes

Rewrite

$$
\begin{aligned}
y_{t+h|t} - y_{t+h|t-1} &= y_t - y_{t-1} - b + (c_1 + c_2 + \cdots + c_h)u_t + (c_{h+1} - c_1)u_{t-1} + (c_{h+2} - c_2)u_{t-2} + \cdots \\
&= (1 + c_1 + c_2 + \cdots + c_h)u_t
\end{aligned}
$$

In the limit, for $h \to \infty$,

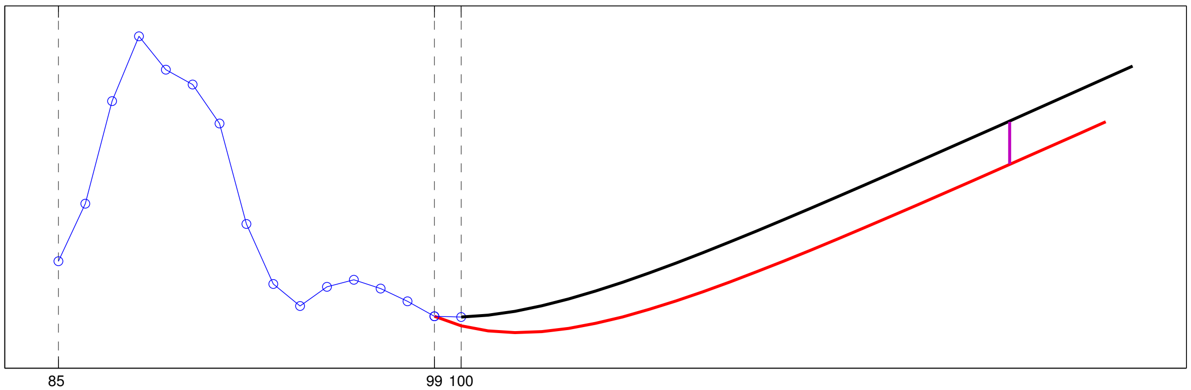
$$
y_{t+h|t} - y_{t+h|t-1} \to (1 + c_1 + c_2 + \cdots)u_t = c(1)u_t
$$

The quantity

$$
1 + c_1 + c_2 + \cdots = c(1)
$$

is called measure of persistence of the process y_t. It is the change in the long-run prediction due to u_t, divided by u_t Equivalently it is the change in long-run prediction due to a shock of unitary value. Note that the persistence of a TS process is zero.

Non-stationary processes



The graph shows, again, the change in the long-run prediction between $t = 99$ and $t = 100$ for a realization of the process

$$y_t = b + y_{t-1} + (1 - 0.8L)^{-1} u_t$$

The purple segment between the two long-run predictions has length

$$(1 + 0.8 + 0.8^2 + \cdots)u_{100} = \frac{1}{1 - 0.8} u_{100} = 5u_{100}$$

5 being in this case the measure of persistence.

Non-stationary processes

A slightly different interpretation of the measure of persistence is the following. Suppose that $u_\tau = 0$ for $\tau > t-1$, i.e. that there are no more shocks after u_{t-1}. Then, for $\tau > t-1$,

The value of y_τ under the assumption that $u_\tau = 0$ for $\tau > t-1 = y_{\tau|t-1}$

In other words, the predicted value of y at $\tau > t-1$, given the values of y up to $t-1$, can be interpreted as the value that y would take at τ if the shock u ceased to hit the process after $t-1$. This is very easy, just go back to the previous slides where we compute $y_{t+h|t-1}$.

Non-stationary processes

Therefore

$$y_{\tau|t} - y_{\tau|t-1}$$

for $\tau > t$, is the difference in the values that y takes at τ, that are caused by u_t.

Thus

$$\lim_{\tau \to \infty} y_{\tau|t} - y_{\tau|t-1}$$

is the long-run contribution of u_t to the level of y.

Non-stationary processes

Important: The persistence for a stationary or a TS process is zero. That is, the long-run prediction does not change with t. If y_t is TS, $y_t = a + bt + x_t$, with x_t stationary and zero mean, then as $h \to \infty$

$$y_{t+h|t} \to a + bt$$

In particular, if $b = 0$ (i.e. y_t is stationary), then the long run prediction is the mean a.

While the shock u_t of a DS process has a permanent effect, namely the change in the long-run prediction, the shock of a stationary or TS process has only a transitory effect. Permanent and transitory shocks will be the subject of the last part of the course.

Non-stationary processes

Exercise: Suppose that x_t is ARMA

$$x_t = \frac{b(L)}{a(L)} u_t$$

with the roots of $b(L)$ and $a(L)$ outside the unit circle. Prove that

$$c(1) = \frac{b(1)}{a(1)} > 0$$

(Hint: Decompose the polynomials using the roots.)

What happens when some root of the MA polynomial (AR polynomial) approaches unity?

Alternative explanations for economic fluctuations

If the model for non-stationarity is TS, then

1. Population growth, accumulation of capital, technical progress, all this secular causes of change are summarized in the (oversimplified) function of time $a + bt$.

2. Shocks to demand (tastes, confidence, interest rate, quantity of money) are represented by the cycle stationary component.

Alternative explanations for economic fluctuations

But if the model for non-stationarity is a DS model, then how do we disentangle shocks to technical knowledge from shocks to demand?

For example,

$$y_t = b + y_{t-1} + (1 - 0.8L)^{-1} v_t$$

Alternative explanations for economic fluctuations

We start with

$$y_t = T_t + C_t$$

where T_t is the component representing population, capital and technical knowledge, whereas C_t is the cycle. Of course we need assumptions.

Assumption 1. T_t is DS, i.e. is non-stationary but

$$T_t - T_{t-1} = b + x_t = b + a(L)u_t$$

is stationary

Assumption 2. The cycle C_t is stationary. If $C_t = c(L)v_t$, then

$$C_t - C_{t-1} = (1 - L)c(L)v_t = d(L)v_t$$

so that $d(1) = 0$.

Assumption 3. The trend T_t and the cycle C_t are orthogonal at all leads and lags. This is equivalent to saying that u_t and v_t are orthogonal at all leads and lags.

Assumption 3. The trend T_t and the cycle C_t are orthogonal at all leads and lags. This is equivalent to saying that u_t and v_t are orthogonal at all leads and lags.

This is not obvious. However it is generally accepted in the literature I am going to discuss. Under Assumption 3, since $T_t - T_{t-1}$ is stationary,

$$\text{var}(y_t - y_{t-1}) = \text{var}(T_t - T_{t-1}) + \text{var}(C_t - C_{t-1})$$

Alternative explanations for economic fluctuations

At the beginning of the 80's the TS model, then traditional in macroeconomic analysis, was tested using results obtained in the 70's by the statisticians Dickey and Fuller.

The DS hypothesis, taken as the null while TS is the alternative, was not rejected for a wide set of macroeconomic indicators, including GNP, Aggregate Consumption, Industrial Production, Inflation, yearly figures with starting dates from 1860 to 1909, ending date 1970. This is the first result in

C.R. Nelson and C.I. Plosser, Trends and random walks in macroeconomic time series, Journal of Monetary Economics, 1982, 10.

This outcome was confirmed for US quarterly data and for data relative to many different countries. Within a few years the TS model was completely superseded by the DS.

Alternative explanations for economic fluctuations

The second main result in Nelson and Plosser (1982) is the following: The autocorrelation function estimated for the GNP implies that "the variation in output changes is dominated by changes in secular component rather than the cyclical component." (p. 155).

This is a big change. Consider again

$$y_t - y_{t-1} = T_t - T_{T-1} + C_t - C_{t-1}$$

If the variance of $T_t - T_{t-1}$ were small as compared with the variance of $C_t - C_{t-1}$, then moving from the traditional TS model to DS would not have important consequences, either interpretive or in terms of economic policy.

But the fact that macroeconomic fluctuations are mainly driven by real shocks, this is an enormous change: There is no longer any room for anticyclical economic policy.

Alternative explanations for economic fluctuations

If you are interested in the topic I urge you to go and read Nelson and Plosser (1982) paper. However, here is some hint.

Suppose that we specify T_t and C_t as

$$T_t = b + u_t, \qquad C_t = v_t$$

so that

$$y_t - y_{t-1} = b + u_t + (v_t - v_{t-1})$$

Alternative explanations for economic fluctuations

Write again the decomposition into trend and cycle

$$y_t - y_{t-1} = b + u_t + (v_t - v_{t-1})$$

A consequence is that the first order autocovariance of $y_t - y_{t-1}$ is $-\sigma_v^2$, the first order autocorrelation being

$$\frac{-\sigma_v^2}{\sigma_u^2 + 2\sigma_v^2}$$

which is negative. Now, this result is inconsistent with the Nelson and Plosser's empirical finding that the first order autocorrelation of the GNP (first difference) is positive.

Objections to Nelson and Plosser.

M.W. Watson, Univariate detrending methods with stochastic trends, Journal of Monetary Economics, 1986, 18.

Watson argues that an ARIMA is only a good approximation to the Wold representation, not **the** Wold representation. Other good approximations can be obtained. He proposed

$$y_t - y_{t-1} = T_t - T_{t-1} + C_t - C_{t-1}$$

with

$$T_t - T_{t-1} = b + u_t, \quad C_t = (1 - \alpha L - \beta L^2)^{-1} v_t$$

so that

$$y_t - y_{t-1} = (b + u_t) + (1 - L)(1 - \alpha L - \beta L^2)^{-1} v_t$$

Here the trend is a random walk while the cycle is AR(2).

Alternative explanations for economic fluctuations

Again Watson's model:

$$y_t - y_{t-1} = (T_t - T_{t-1}) + (C_t - C_{t-1}) = (b + u_t) + (1 - L)(1 - \alpha L - \beta L^2)^{-1} v_t$$

that is

$$(1 - \alpha L - \beta L^2)(y_t - y_{t-1}) = b(1 - \alpha - \beta) + (1 - \alpha L - \beta L^2)u_t + (1 - L)v_t$$

This is called an Unobserved Components model, the components being T_t and C_t. The parameters $b,\ \alpha,\ \beta,\ \sigma_u^2,\ \sigma_v^2$ are called hidden parameters.

We can prove that

$$(1 - \alpha L - \beta L^2)(y_t - y_{t-1}) = c + (1 - AL - BL^2)w_t$$

The latter is an ARIMA(2,1,2) for the observable $y_t - y_{t-1}$.

Alternative explanations for economic fluctuations

Write again the two representations for $y_t - y_{t-1}$ (dropping the constant terms for simplicity:

$$\begin{aligned} (1-\alpha L-\beta L^2)(y_t-y_{t-1}) &= (1-\alpha L-\beta L^2)u_t+(1-L)v_t && (*) \\ (1-\alpha L-\beta L^2)(y_t-y_{t-1}) &= (1-AL-BL^2)w_t && (**) \end{aligned}$$

Equating the autocovariances of the right hand sides we obtain the three equations:

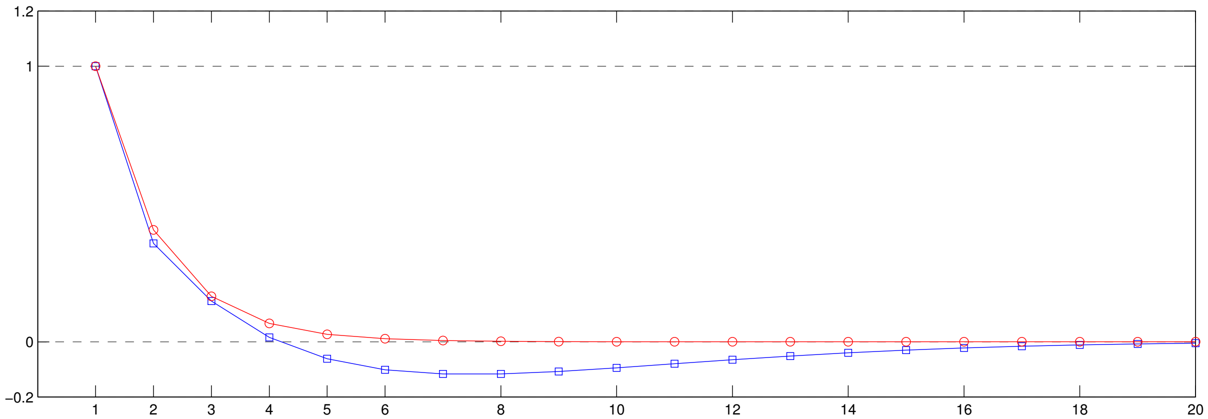
$$\begin{aligned} (1+\alpha^2+\beta^2)\sigma_u^2+2\sigma_v^2 &= (1+A^2+B^2)\sigma_w^2 \\ (-\alpha+\beta\alpha)\sigma_u^2-\sigma_v^2 &= (-A+BA)\sigma_w^2 \\ -\beta\sigma_u^2 &= -B\sigma_w^2 \end{aligned} \qquad \text{(S)}$$

An idea of the estimation strategy can be as follows: Estimate the ARIMA(2,1,2) in $(**)$, thus determining the parameters α and β, then use system (S) to compute σ_u^2 and σ_v^2. Watson shows that the solution is unique (not easy).

Alternative explanations for economic fluctuations

Lastly we obtain the coefficients of the Wold representation implicit in the UC model:

$$y_t - y_{t-1} = \frac{1 - AL - AL^2}{1 - \alpha L - \beta L^2} w_t = (1 + D_1 L + D_2 L^2 + \cdots) w_t$$



The figure has the plot of the sequence D_k (blue line) together with the coefficients of the Wold representation implicit in an AR(1) model estimated for $y_t - y_{t-1}$:

$$y_t - y_{t-1} = \frac{1}{1 - \delta L} a_t = (1 + \delta L + \delta^2 L^2 + \cdots) a_t$$

Results:

1. The variance of a_t, the AR(1) residual, and of w_t, the residual implicit in the UC model are not significantly different. In other words, imposing the UC structure does not worsen the fit.

2. Contrary to Nelson and Plosser's finding, the variance of the trend, i.e. the variance of $T_t - T_{t-1} = b + u_t$, is smaller as compared to the variance of the cycle.

Alternative explanations for economic fluctuations

Another objection to Nelson and Plosser. The UC model presented above has a random walk trend and an AR(2) cyclical component:

$$T_t = T_{t-1} + b + u_t, \quad C_t = (1 - \alpha L - \beta L^2)^{-1} v_t$$

Now remember that the trend should represent mainly the change in productivity due to technical progress. A random walk implies that there is no correlation between the change in productivity at time t and the change at time $t-1$. But this is not reasonable.

Processes like learning by doing within a firm, or diffusion of technical change among firms, imply a smooth process like

$$T_t - T_{t-1} = (p_0 + p_1 L + p_2 L^2 + \cdots) u_t$$

The shock u_t takes time to be entirely transferred into productivity increase. The process is described by the coefficients p_k. We can assume that $p_k \geq 0$ and that

$$p_0 + p_1 + p_2 + \cdots = 1$$

Alternative explanations for economic fluctuations

It is possible to prove that a process like

$$T_t - T_{t-1} = (p_0 + p_1 L + p_2 L^2 + \cdots)u_t$$

can have a very small variance. An intuition is provided by the following example

$$T_t - T_{t-1} = (1-\alpha)(1 + \alpha L + \alpha^2 L^2 + \cdots)u_t$$

The coefficients sum to unity. Moreover, given σ_u^2, as $\alpha \to 0$,

$$\text{var}(T_t - T_{t-1}) = \frac{(1-\alpha)^2}{1-\alpha^2}\sigma_u^2 = \frac{1-\alpha}{1+\alpha}\sigma_u^2 \to 0$$

For this kind of objection see

D. Quah, The Relative Importance of Permanent and Transitory Components: Identification and Some Theoretical Bounds, Econometrica, 1992, 60.

M. Lippi and L. Reichlin, Diffusion of Technical Change and the Identification of the Trend Component in Real GNP, Review of Economic Studies, 1994, 61.

Alternative explanations for economic fluctuations

A multivariate approach to the problem of determining the relative importance of trend and cycle in explaining macroeconomic fluctuations.

We have seen that the results obtained by Nelson and Plosser and other authors depend very much on the model they start with: an ARIMA, UC models, which in turn depend on the way we specify the trend and the cycle components. Now I will discuss the paper

O.J. Blanchard and D. Quah, The dynamic effect of aggregate demand and supply disturbances, American Economic Review, 1989, 79,

in which the assessment on trend and cycle depends only on the data.

Consider the stochastic stationary vector

$$z_t = \begin{pmatrix} \Delta y_t \\ U_t \end{pmatrix}$$

in which Δ denoted $1 - L$, so that the first component of z_t is $y_t - y_{t-1}$, while U_t is the unemployment rate.

Alternative explanations for economic fluctuations

BQ start by estimating a VAR for the vector z_t:

$$(I - A_1 L - A_2 L^2 - \cdots - ApL^p) \begin{pmatrix} \Delta y_t \\ U_t \end{pmatrix} = A(L) \begin{pmatrix} \Delta y_t \\ U_t \end{pmatrix} = \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix}$$

This means:

$$\begin{aligned} \Delta y_t &= [a_{11,1}\Delta y_{t-1} + a_{11,2}\Delta y_{t-2} + \cdots + a_{11,p}\Delta y_{t-p}] + [a_{12,1}U_{t-1} + a_{12,2}U_{t-2} + \cdots + a_{12,p}U_{t-p}] + u_{1t} \\ U_t &= [a_{21,1}\Delta y_{t-1} + a_{21,2}\Delta y_{t-2} + \cdots + a_{21,p}\Delta y_{t-p}] + [a_{22,1}U_{t-1} + a_{22,2}U_{t-2} + \cdots + a_{22,p}U_{t-p}] + u_{2t} \end{aligned}$$

The result of the estimation is $A_1,\ A_2,\ \cdots,\ A_p$, plus the matrix

$$\Sigma_u = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

i.e. the variance-covariance matrix of the residual vector.

Alternative explanations for economic fluctuations

Start again with the estimated VAR:

$$A(L)\begin{pmatrix}\Delta y_t\\ U_t\end{pmatrix}=\begin{pmatrix}u_{1t}\\ u_{2t}\end{pmatrix}$$

Denote by $B(L)$ the inverse $A(L)^{-1}$:

$$\begin{pmatrix}\Delta y_t\\ U_t\end{pmatrix}=A(L)^{-1}\begin{pmatrix}u_{1t}\\ u_{2t}\end{pmatrix}=B(L)\begin{pmatrix}u_{1t}\\ u_{2t}\end{pmatrix}=(I+B_1L+B_2L^2+\cdots)\begin{pmatrix}u_{1t}\\ u_{2t}\end{pmatrix}$$
$$=\begin{pmatrix}b_{11}(L) & b_{12}(L)\\ b_{21}(L) & b_{22}(L)\end{pmatrix}\begin{pmatrix}u_{1t}\\ u_{2t}\end{pmatrix}$$

That is:

$$\begin{aligned}\Delta y_t &= b_{11}(L)u_{1t}+b_{12}(L)u_{2t}=[u_{1t}+b_{11,1}u_{1,t-1}+b_{11,2}u_{1,t-2}+\cdots]+[\quad b_{12,1}u_{2,t-1}+b_{12,2}u_{2,t-2}+\cdots]\\ U_t &= b_{21}(L)u_{1t}+b_{22}(L)u_{2t}=[\quad b_{21,1}u_{1,t-1}+b_{21,2}u_{1,t-2}+\cdots]+[u_{2t}+b_{22,1}u_{2,t-1}+b_{22,2}u_{2,t-2}+\cdots]\end{aligned}$$

This is the Wold representation.

Alternative explanations for economic fluctuations

Again the Wold representation:

$$
\begin{aligned}
\Delta y_t &= b_{11}(L)u_{1t} + b_{12}(L)u_{2t} = [u_{1t} + b_{11,1}u_{1,t-1} + b_{11,2}u_{1,t-2} + \cdots] + [\qquad b_{12,1}u_{2,t-1} + b_{12,2}u_{2,t-2} + \cdots] \\
U_t &= b_{21}(L)u_{1t} + b_{22}(L)u_{2t} = [\qquad b_{21,1}u_{1,t-1} + b_{21,2}u_{1,t-2} + \cdots] + [u_{2t} + b_{22,1}u_{2,t-1} + b_{22,2}u_{2,t-2} + \cdots]
\end{aligned}
$$

You see here that Δy_t and U_t are driven by u_{1t} and u_{2t}. Of course saying that, for example, u_{1t} is the shock to technical knowledge while u_{2t} is the shock to demand does not make any sense. In the first place, u_{1t} and u_{2t} are not orthogonal in general.

However, we can transform the vector $(u_{1t}\ u_{2t})$ in such a way that the above interpretation is possible.

Alternative explanations for economic fluctuations

First we transform the shocks u_{1t} and u_{2t} into v_{1t} and v_{2t} with $v_{1t} \perp v_{2t}$. This is easy. Consider the regression:

$$u_{2t} = \alpha u_{1t} + s_t, \quad \text{with} \quad \alpha = \sigma_{12}/\sigma_{11}$$

Of course $s_t \perp u_{1t}$.

We have

$$\begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} u_{1t} \\ s_t \end{pmatrix}$$

and

$$\begin{pmatrix} \Delta y_t \\ U_t \end{pmatrix} = \begin{pmatrix} b_{11}(L) & b_{12}(L) \\ b_{21}(L) & b_{22}(L) \end{pmatrix} \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix} = \left[\begin{pmatrix} b_{11}(L) & b_{12}(L) \\ b_{21}(L) & b_{22}(L) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \right] \begin{pmatrix} u_{1t} \\ s_t \end{pmatrix} = \begin{pmatrix} c_{11}(L) & c_{12}(L) \\ c_{21}(L) & c_{22}(L) \end{pmatrix} \begin{pmatrix} v_{1t} \\ v_{2t} \end{pmatrix}$$

Alternative explanations for economic fluctuations

Thus now we have

$$\begin{pmatrix} \Delta y_t \\ U_t \end{pmatrix} = \begin{pmatrix} c_{11}(L) & c_{12}(L) \\ c_{21}(L) & c_{22}(L) \end{pmatrix} \begin{pmatrix} v_{1t} \\ v_{2t} \end{pmatrix}$$

in which Δy_t and U_t are driven by orthogonal shocks. We further transform the representation in such a way that the shocks are normalized, i.e. have unit variance:

$$\begin{pmatrix} \Delta y_t \\ U_t \end{pmatrix} = \begin{pmatrix} c_{11}(L) & c_{12}(L) \\ c_{21}(L) & c_{22}(L) \end{pmatrix} \begin{pmatrix} v_{1t} \\ v_{2t} \end{pmatrix} = \left[\begin{pmatrix} c_{11}(L) & c_{12}(L) \\ c_{21}(L) & c_{22}(L) \end{pmatrix} \begin{pmatrix} \sigma_{v_1} & 0 \\ 0 & \sigma_{v_2} \end{pmatrix} \right] \begin{pmatrix} \sigma_{v_1}^{-1} & 0 \\ 0 & \sigma_{v_2}^{-1} \end{pmatrix} \begin{pmatrix} v_{1t} \\ v_{2t} \end{pmatrix} = \begin{pmatrix} d_{11}(L) & d_{12}(L) \\ d_{21}(L) & d_{22}(L) \end{pmatrix} \begin{pmatrix} w_{1t} \\ w_{2t} \end{pmatrix}$$

with obvious definitions.

Summing up

$$\begin{pmatrix} \Delta y_t \\ U_t \end{pmatrix} = B(L)u_t = C(L)v_t = D(L)w_t$$

Going over the definitions you see that

$$B(0) = I_2, \quad C(0) = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \quad D(0) = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \sigma_{v_1} & 0 \\ 0 & \sigma_{v_2} \end{pmatrix} = \begin{pmatrix} \sigma_{v_1} & 0 \\ \alpha\sigma_{v_1} & \sigma_{v_2} \end{pmatrix}$$

Considering the last representation,

$$\begin{pmatrix} \Delta y_t \\ U_t \end{pmatrix} = \begin{pmatrix} d_{11}(L) & d_{12}(L) \\ d_{21}(L) & d_{22}(L) \end{pmatrix} \begin{pmatrix} w_{1t} \\ w_{2t} \end{pmatrix}, \quad D(0) = \begin{pmatrix} \sigma_{v_1} & 0 \\ \alpha\sigma_{v_1} & \sigma_{v_2} \end{pmatrix}$$

so that

$$\begin{aligned}
\Delta y_t &= [d_{11,0}w_{1t} + d_{11,1}w_{1,t-1} + d_{11,2}w_{1,t-2} + \cdots] + [d_{12,0}w_{2t} + d_{12,1}w_{2,t-1} + d_{12,2}w_{2,t-2} + \cdots] \\
&= [\ \sigma_{v_1}w_{1t} + d_{11,1}w_{1,t-1} + d_{11,2}w_{1,t-2} + \cdots] + [\qquad\qquad d_{12,1}w_{2,t-1} + d_{12,2}w_{2,t-2} + \cdots] \\
U_t &= [d_{21,0}w_{1t} + d_{21,1}w_{1,t-1} + d_{21,2}w_{1,t-2} + \cdots] + [d_{22,0}w_{2t} + d_{22,1}w_{2,t-1} + d_{22,2}w_{2,t-2} + \cdots] \\
&= [\alpha\sigma_{v_1}w_{1t} + d_{21,1}w_{1,t-1} + d_{21,2}w_{1,t-2} + \cdots] + [\ \sigma_{v_2}w_{2t} + d_{22,1}w_{2,t-1} + d_{22,2}w_{2,t-2} + \cdots]
\end{aligned}$$

$$\begin{aligned} \Delta y_t &= [\ \sigma_{v_1} w_{1t} + d_{11,1} w_{1,t-1} + d_{11,2} w_{1,t-2} + \cdots] + [\ d_{12,1} w_{2,t-1} + d_{12,2} w_{2,t-2} + \cdots] \\ U_t &= [\alpha \sigma_{v_1} w_{1t} + d_{21,1} w_{1,t-1} + d_{21,2} w_{1,t-2} + \cdots] + [\ \sigma_{v_2} w_{2t} + d_{22,1} w_{2,t-1} + d_{22,2} w_{2,t-2} + \cdots] \end{aligned}$$

Now suppose that we believe that the shock to demand impacts the GNP within the quarter, whereas the impact of the shock to technology occurs with a lag of one quarter. Then w_{1t} and w_{2t} can be interpreted as the shock to demand and the shock to technology respectively, and the representation above is structural.

The assumptions

1. Orthogonality of the shocks to demand and technology,
2. Normalization of the shocks.
3. Demand shocks impact within the quarter, technology shocks with delay,

are sufficient for identification of the shocks w_t and the matrix $D(L)$.

Alternative explanations for economic fluctuations

Assuming that the shocks are orthogonal and that their impact on the macroeconomic variables occur in different quarters has been the most usual identification assumption in Structural VAR analysis for some time, following C. Sims, Macroeconomics and reality, Econometrica, 1980, 48.

However, identification condition 3 is not the one we want to assume. Remember that

$$y_{t-1} = T_t + C_t$$

where T_t is DS and C_t is stationary. Now we want to elaborate on this.

Alternative explanations for economic fluctuations

Consider again

$$\begin{pmatrix} \Delta y_t \\ U_t \end{pmatrix} = B(L)u_t = C(L)v_t = D(L)w_t$$

This is where we are. Remember that the components of w_t are orthonormal.

Obviously we can obtain infinitely many other reprentations in which the white noise is orthonormal:

$$\begin{pmatrix} \Delta y_t \\ U_t \end{pmatrix} = D(L)M^{-1}Mw_t = F(L)z_t$$

provided that $z_t = Mw_t$ is orthonormal

Alternative explanations for economic fluctuations

So we want that

$$z_t = Mw_t = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} w_{1t} \\ w_{2t} \end{pmatrix} = \begin{pmatrix} m_{11}w_{1t} + m_{12}w_{2t} \\ m_{21}w_{1t} + m_{22}w_{2t} \end{pmatrix}$$

is orthonormal.

Using the assumption that w_t is orthonormal, we find the conditions

$$m_{11}^2 + m_{12}^2 = 1, \quad m_{21}^2 + m_{22}^2 = 1, \quad m_{11}m_{21} + m_{12}m_{22} = 0$$

This is equivalent to

$$MM' = I, \quad \text{that is} \quad M^{-1} = M'$$

M is called a unitary matrix.

2-dimensional unitary matrices have the representation

$$M = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$$

that is a symmetry plus a rotation of angle θ.

Back to

$$\begin{pmatrix} \Delta y_t \\ U_t \end{pmatrix} = D(L) M^{-1} M w_t = F(L) z_t$$

We see that the symmetry has only effect on the sign of the components of w_t.

This can be dealt with separately. So let us concentrate on the rotation.

Alternative explanations for economic fluctuations

Again

$$\begin{pmatrix} \Delta y_t \\ U_t \end{pmatrix} = \begin{pmatrix} d_{11}(L) & d_{12}(L) \\ d_{21}(L) & d_{22}(L) \end{pmatrix} \begin{pmatrix} w_{1t} \\ w_{2t} \end{pmatrix}$$

Insert the rotation

$$\begin{pmatrix} \Delta y_t \\ U_t \end{pmatrix} = [D(L)M^{-1}][Mw_t] = \left[\begin{pmatrix} d_{11}(L) & d_{12}(L) \\ d_{21}(L) & d_{22}(L) \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right] \left[\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} w_{1t} \\ w_{2t} \end{pmatrix} \right]$$

$$= \begin{pmatrix} d_{11}(L)\cos\theta + d_{12}(L)\sin\theta & -d_{11}(L)\sin\theta + d_{12}(L)\cos\theta \\ d_{21}(L)\cos\theta + d_{22}(L)\sin\theta & -d_{21}(L)\sin\theta + d_{22}(L)\cos\theta \end{pmatrix} \begin{pmatrix} z^{\theta}_{1t} \\ z^{\theta}_{2t} \end{pmatrix}$$

In particular

$$\Delta y_t = [d_{11}(L)\cos\theta + d_{12}(L)\sin\theta] z^{\theta}_{1t} + [-d_{11}(L)\sin\theta + d_{12}(L)\cos\theta] z^{\theta}_{2t}$$

Alternative explanations for economic fluctuations

Again:

$$\Delta y_t = [d_{11}(L)\cos\theta + d_{12}(L)\sin\theta] z^{\theta}_{1t} + [-d_{11}(L)\sin\theta + d_{12}(L)\cos\theta] z^{\theta}_{2t}$$

But

$$\Delta y_t = \Delta T_t + \Delta C_t = d(L)u_t + (1-L)c(L)v_t$$

But then if, say, z^{θ}_{2t} is the shock to the cycle we must have

$$-d_{11}(1)\sin\theta + d_{12}(1)\cos\theta = 0$$

that is

$$\tan\theta = \frac{d_{12}(1)}{d_{11}(1)}$$

It only remains to decide which of the matrices

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$$

must be chosen.

But this will be done by elementary considerations. For example, we want that positive shocks to technology and the cycle have a positive first impact on Δy_t.

Alternative explanations for economic fluctuations

In conclusion, we have identified

$$\begin{pmatrix} \Delta y_t \\ U_t \end{pmatrix} = F(L) z_t$$

with $F_{12}(1) = 0$ so that z_{2t} has the interpretation as a shock to the business cycle while z_{1t} is the technology shock. The analysis presented above is known as Structural VAR analysis (SVAR).

Blanchard and Quah found that the variance of the cycle component was bigger than that of the trend component, a result that is in contrast with Nelson and Plosser's finding.

Our last topic is the impulse-response functions of a SVAR. Suppose that

$$z_{1t} = \begin{cases} 1 \text{ for } t = 0 \\ 0 \text{ for all } t \neq 0 \end{cases}$$

and that $z_{2t} = 0$ for all t. Then, as you easily see, the variables Δy_t and U_t would follow the paths

Time	$\cdots$	-1	0	1	2	3	$\cdots$
Δy_t	$\cdots$	0	$f_{11,0}$	$f_{11,1}$	$f_{11,2}$	$f_{11,3}$	$\cdots$
U_t	$\cdots$	0	$f_{21,0}$	$f_{21,1}$	$f_{21,2}$	$f_{21,3}$	$\cdots$

The sequences above are called the impulse-response functions of Δy_t and U_t, respectively, to a unitary shock to technology. The definition for the shock to the cycle is obvious.

Alternative explanations for economic fluctuations

The impulse-response function of Δy_t to the technology shock is

$$f_{11,0} \quad f_{11,1} \quad f_{11,2} \quad \cdots$$

and analogously for the shock to the cycle. The impulse-response functions of y_t to the technology shock is obtained by cumulating:

$$f_{11,0} \quad f_{11,0} + f_{11,1} \quad f_{11,0} + f_{11,1} + f_{11,2} \quad \cdots$$

Taking a look at Blanchard and Quah's estimated impulse-response functions is recommended.