2.5 Math 2. Fourier series

Starting with

$$e^{ik\theta} = \cos k\theta + i\sin k\theta,\ \ \cos k\theta = \frac{e^{ik\theta}+e^{-ik\theta}}{2},\ \ \cos k\theta = \frac{e^{ik\theta}-e^{-ik\theta}}{2i},$$

(see Exercise 1.20) the reader will easily check that
(a) the functions

$$\frac{1}{2},\ \ \cos\theta,\ \ \sin\theta,\ \ \cos 2\theta,\ \ \sin 2\theta,\ \ \ldots \tag{2.67}$$

form an orthogonal system of real functions,
(b) their modulus are $\sqrt{\pi/2}$ for the first, $\sqrt{\pi}$ for all the others,
(c) the subspace of $L^2([-\pi\ \pi])$ spanned by the functions (2.67) and the subspace H^e spanned by the functions

$$\ldots,\ \ e^{i2\theta},\ \ e^{i\theta},\ \ 1,\ \ e^{-i\theta},\ \ e^{-i2\theta},\ \ \ldots$$

coincide,
(d) for every function $f \in L^2([-\pi\ \pi])$,

$$f(\theta) = \frac{b_{f0}}{2} + \sum_{k=1}^{\infty}\left(b_{fk}\cos k\theta + c_{fk}\sin k\theta\right) + R_f(\theta), \tag{2.68}$$

(mean-square convergence) where

$$\begin{aligned} b_{fk} &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos k\theta d\theta,\ \ k = 0, 1, \ldots \\ c_{fh} &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin h\theta d\theta,\ \ h = 1, 2, \ldots \end{aligned} \tag{2.69}$$

Comparison of (2.69) and (2.54) shows that, for $k = 0, 1, \ldots,$

$$\begin{aligned} b_{fk} &= a_{f,-k} + a_{fk},\ \ \ c_{fk} = i(a_{f,-k} - a_{fk}) \\ a_{fk} &= \frac{b_{fk}\ + ic_{fk}}{2},\ \ a_{f,-k} = \frac{b_{fk} - ic_{fk}}{2}. \end{aligned} \tag{2.70}$$

Equation (2.68) is just another version of (2.55), in which the same orthogonal projection is obtained with a different orthogonal family, the residual being the same. What Proposition 2.9 proves is that R_f is a.e. zero, so that $H^e = L^2([-\pi\ \pi])$. Proofs of Proposition 2.9 can be found in many textbooks (see e.g.

[15], p. 328, Theorem 11.40). A proof requiring only elementary analysis can be found in [16], pp. 11-12.

In conclusion, any function f belonging to $L^2([-\pi\ \pi])$ is the mean-square limit of its Fourier expansion, the latter taking either the complex form

$$f(\theta) = \sum_{k=-\infty}^{\infty} a_{fk} e^{-ik\theta}, \tag{2.71}$$

or the real form

$$f(\theta) = \frac{b_{f0}}{2} + \sum_{k=1}^{\infty} \left(b_{fk} \cos k\theta + c_{fk} \sin k\theta \right). \tag{2.72}$$

Note that in (2.72) the coefficients are in general complex (see Exercise 2.18). The functions $\cos k\theta$ and $\sin k\theta$, which are the buiding blocks by means of which all functions are reconstructed in Fourier analysis, are called *harmonic oscillations*, while *complex harmonic oscillations* is used for the functions $e^{-ik\theta}$.

Exercise 2.18 Show that
(1) If f is real then $a_k = \overline{a}_{-k}$. As a consequence the coefficients b_k and c_k are real. Use (2.70).
(2) If f is even, i.e. $f(\theta) = f(-\theta)$, then $c_k = 0$ for all $k \geq 1$.
(3) If f is odd, i.e. $f(\theta) = -f(-\theta)$, then $b_k = 0$ for all $k \geq 0$.
For (2) and (3) you can use the fact that in (2.72) the function f is written as an even function plus an odd function.

Some comments are in order. Firstly, note that Proposition 2.9 proves more than we need. Its assumption requires that g be integrable and that

$$\int_{-\pi}^{\pi} g(\theta) e^{ik\theta} d\theta = 0$$

for all $k \in \mathbb{Z}$, or, equivalently,

$$\int_{-\pi}^{\pi} g(\theta) \cos k\theta d\theta = \int_{-\pi}^{\pi} g(\theta) \sin k\theta d\theta = 0,$$

whereas for our purpose $g \in L^2([-\pi\ \pi])$ would be sufficient.

Secondly, the reader should keep in mind that Proposition 2.9 allows the conclusion that the Fourier expansions (2.71) or (2.72) converge in *mean square.* It does not imply any further consequence. In particular:

(i) Proposition 2.9 says nothing on the conditions under which the Fourier series of a function of $L^2([-\pi\ \pi])$ converges everywhere, almost everywhere or at any given point in $[-\pi\ \pi]$.
(ii) It says nothing on the convergence of the Fourier series of a continuous function.
These are central and very difficult issues of the theory of Fourier series, which is in itself an important branch of analysis. For our purposes we only need, in addition to Proposition 2.9, some relatively elementary results that require the notion of function of bounded variation.

Definition 2.9 Let f be a function with domain in $[-\pi\ \pi]$. Consider the partition $P = (\theta_1, \theta_2, \ldots, \theta_n)$, with

$$-\pi = \theta_1 < \theta_2 < \cdots < \theta_n = \pi$$

and define

$$S(P) = \sum_{k=1}^{n-1} |f(\theta_{k+1}) - f(\theta_k)|.$$

The function f is of *bounded variation* if the set of all non-negative real numbers $S(P)$, P ranging in the set of all possible partitions (for any length n), is bounded.

An easy exercise shows that a function of bounded variation must be bounded. The function taking value 0 for $\theta < 0$ and 1 for $\theta \geq 0$ is of course of bounded variation, so that bounded variation does not imply continuity. Nor does continuity imply bounded variation. To see this, consider the following functions:

$$f_1(\theta) = \begin{cases} 0 \text{ if } \theta = 0 \\ \sin(1/\theta) \text{ if } \theta \neq 0, \end{cases} \qquad f_2(\theta) = \theta f_1(\theta), \quad f_3(\theta) = \theta^2 f_1(\theta).$$

The reader should be able to show that f_2 is continuous at $\theta = 0$ but not of bounded variation, while f_3, though oscillating infinitely many times as θ approaches 0, is of bounded variation.

Bounded variation has important consequences on the convergence of Fourier series.

Proposition 2.18 Suppose that f is integrable in $[-\pi\ \pi]$ and is of bounded variation. Then the partial Fourier sums

$$S_{f,m}(\theta) = \sum_{k=-m}^{m} a_{fk} e^{-ik\theta} = \frac{b_{f0}}{2} + \sum_{k=1}^{m} (b_{fk} \cos k\theta + c_{fk} \sin k\theta)$$

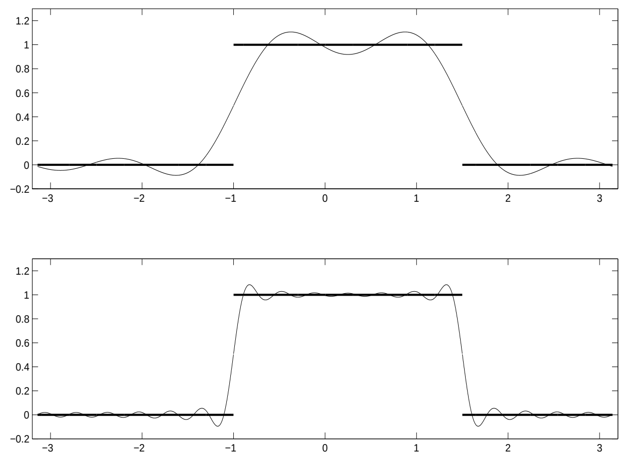

FIGURE 2.4: Fourier approximation of the function (2.73), $m = 4$ and $m = 18$

are uniformly bounded, i.e. there exists a real M such that

$$\left|S_{f,m}(\theta)\right| < M,$$

for all $m \in \mathbb{N}$ and $\theta \in [-\pi\ \pi]$. See [16], p. 90, Theorem 3.7.

To provide the reader with an idea of the complications of Fourier theory let us only mention that the conclusion of Proposition 2.18 cannot be obtained dropping the assumption of bounded variation, even when the function f is continuous (for the existence of continuous functions whose Fourier sums diverge at a point see [16], Chapter VIII, p. 298). Here we must recall that in Fourier theory the term "continuous" is reserved to functions f that are continuous in the usual sense and fulfill the condition $f(-\pi) = f(\pi)$. The reason for this is obvious: the approximating functions have period 2π, hence $S_{f,m}(-\pi) = S_{f,m}(\pi)$, so that $f(-\pi) \neq f(\pi)$ is equivalent to a discontinuity.

Another important theorem requires that we firstly recall that a function of bounded variation can be expressed as the difference of monotonically increasing functions (see [3], p. 132). Therefore, for all θ internal to $[-\pi\ \pi]$, the left and right limits of f exist. Set $f(\theta^-) = \lim_{\theta\to\theta^-} f(\theta)$ and $f(\theta^+) = \lim_{\theta\to\theta^+} f(\theta)$. For $-\pi$ and π we define $f(-\pi^-) = f(\pi^-)$ and $f(\pi^+) = f(-\pi^+)$.

Proposition 2.19 Suppose that f is integrable in $[-\pi\ \pi]$ and is of bounded variation. Then:
(i) The Fourier sums converge to the average

$$\frac{f(\theta^+) + f(\theta^-)}{2}.$$

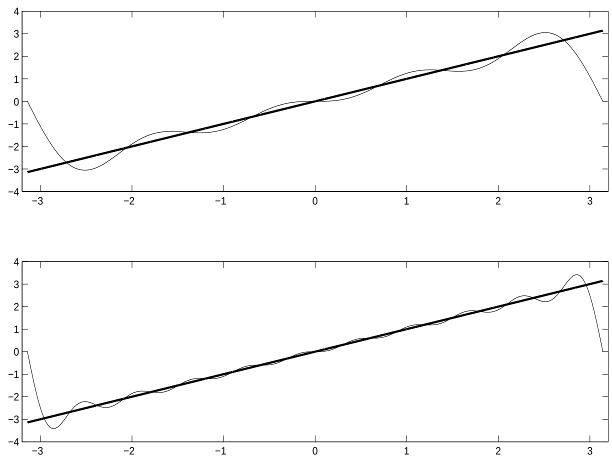

FIGURE 2.5: Fourier approximation of $g(\theta) = \theta$, $m = 4$ and $m = 10$

(ii) In particular, if θ is a point of continuity of f the Fourier sums of f converge to $f(\theta)$.
(iii) If f is continuous at every point of the closed interval $[a\ b] \subset [-\pi\ \pi]$ (meaning, in particular, that $f(a^-) = f(a^+)$ and $f(b^-) = f(b^+)$) then the Fourier sums converge to f uniformly in $[a\ b]$. In particular, if f is continuous on $[-\pi\ \pi]$, (with the condition, let us insist, $f(-\pi) = f(\pi)$), then the Fourier sums converge uniformly to f in $[-\pi\ \pi]$. See [16], p. 57, Theorem 8.1.

Example 2.11 Figure 2.4 shows the partial Fourier sums for the function

$$f(\theta) = \begin{cases} 1 \text{ if } -1 < \theta < 1.5 \\ 0 \text{ otherwise,} \end{cases} \tag{2.73}$$

corresponding to $m = 4$, upper graph, and $m = 18$, lower graph. The function f, thicker line, has two discontinuities at -1 and 1.5 and is obviously of bounded variation. By Proposition 2.19, (i), the sums converge to 0.5 at -1 and 1.5, as clearly visible in the graphs. Convergence to f is uniform in any interval $[a\ b]$ not containing a or b. The same considerations apply to Figure 2.5, which refers to $g(\theta) = \theta$. Note that in this case the discontinuity occurs at π and $-\pi$, where the Fourier sums converge, according to Proposition 2.19, (i), to 0, which is the average between $g(\pi^-) = \pi$ and $g(\pi^+) = g(-\pi^+) = -\pi$.

We cannot conclude this section without mentioning the role played by the Fourier series in the analysis of periodic phenomena. This will be done by a short and elementary reference to the theory of sound. Assume that F is defined on the

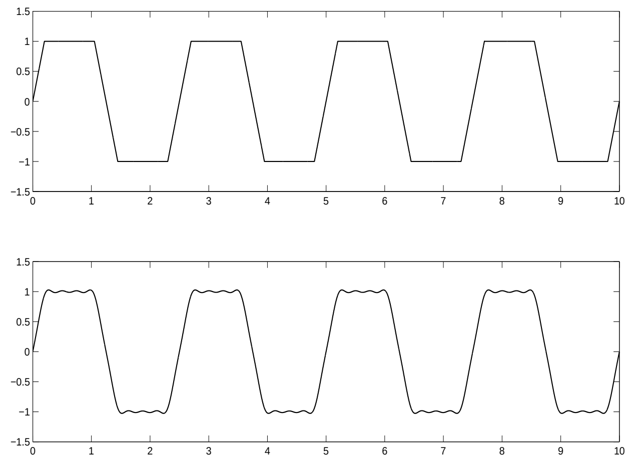

FIGURE 2.6: A vibration and its Fourier approximation, $m = 10$

whole real line $\mathbb{R}$, is periodic of period 2π and its restriction to $[-\pi\ \pi]$ belongs to $L^2([-\pi\ \pi])$. We write, for all $x \in \mathbb{R}$,

$$F(x) = \frac{b_{F0}}{2} + b_{F1}\cos x + c_{F1}\sin x + b_{F2}\cos 2x + c_{F2}\sin 2x + \cdots,$$

by this meaning that all the properties of the approximation in $[-\pi\ \pi]$ (mean-square convergence, pointwise convergence, convergence at a point, uniform convergence) are extended to any interval $[k\pi\ (k+2)\pi]$. Defining $F_\phi(x) = F(\phi x)$ we obtain a periodic function of frequency ϕ and period $2pi/\phi$. The Fourier series of F_ϕ is obviously

$$F_\phi(x) = \frac{b_{F0}}{2} + b_{F1}\cos\phi x + c_{F1}\sin\phi x + b_{F2}\cos 2\phi x + c_{F2}\sin 2\phi x + \cdots,$$

the coefficients being the same as those of F, while the harmonic oscillations are $\cos k\phi x$ and $\sin k\phi x$.

Now consider Figure (2.6). Assume that the variable x is time and that the function drawn in the upper graph represents the measure of air-pressure exerted on a human tympanic membrane (rescaled, so that the average is zero). Then:
(I) The human feeling of a *sound*, as opposed to noise, is the result of the periodicity of the function.
(II) The frequency of the vibration determines the *pitch* of the sound, high or low pitch corresponding to high or low frequency. Traditional western music makes use only of a discrete set of frequencies, the *notes*. Think of the keyboard of a piano,

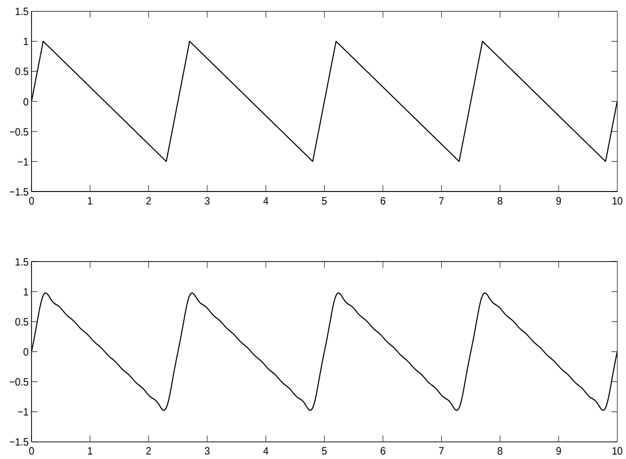

FIGURE 2.7: A vibration and its Fourier approximation, $m = 6$

each key corresponding to a note. However, a violin or a human voice can produce a continuum of pitches. Assuming that a unit on the horizontal axis in Figure 2.6 represents one thousandth of a second, the frequency (oscillations per unit of time, not angular frequency) of the vibration is 400Hz, 1 Hz being 1 cycle per second.

(III) The *amplitude* of the vibration can be measured by the difference between maxima and minima and can be changed by mere multiplication, the amplitude of $aF(\phi x)$ being a times the amplitude of $F(x)$. The amplitude determines the feeling of greater or lesser *loudness* of the sound.

(IV) Lastly, the vibration has a shape. In Figure 2.7 we see another vibration with the same frequency, thus producing the same note, with the same loudness, but with a different shape. The quality of the sound produced by the shape is called *timbre*. Different shapes of vibrations of the same frequency make the difference between different instruments, or different human voices, playing the same note. For example, if Luciano Pavarotti and Lou Reed sing the same note they produce vibrations of the same frequency, possibly with the same loudness, but definitely with different shapes and therefore different timbres. The same can be said when, say, a clarinet and a trumpet play the same note.

(V) Different timbres, i.e. different shapes of the vibration, correspond to different sequences of coefficients in the Fourier series. In particular, the vibrations drawn in Figures 2.6 and 2.7 are both odd functions, so that all the coefficients b vanish and the series contain only the sines' terms. The coefficients c are drawn in Figure 2.8, the bullets and the circles representing the c coefficients of the functions in 2.6 and 2.7, respectively.

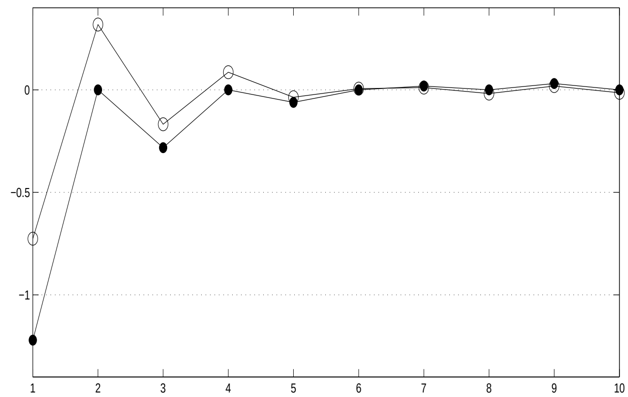

FIGURE 2.8: The first 10 Fourier coefficients for the functions in Figures 2.6 (bullets) and 2.7 (circles)

(VI) The lower graphs in Figures 2.6 and 2.7 show Fourier approximations of the vibrations. They are obtained with $m = 10$ and $m = 6$ respectively. Thus if we are able to produce by an electronic device simple vibrations as the harmonic oscillations, then we can "reconstruct" the vibrations in the upper graphs with an accuracy depending on how far we push the number m. More in general, electronic production of harmonic oscillations of any frequency and amplitude allows obtaining artificial sound of any timbre, pitch and loudness.

We conclude here our illustration. The reader should consider it (points (IV), (V) and (VI) in particular) as no more than a highly stylized account and an inducement to further reading.

Summary. Every function f of $L^2([-\pi\ \pi])$ is the limit in mean square of harmonic oscillations. If the function f is of bounded variation then the approximating functions are uniformly bounded, converge at every point of continuity of f and converge uniformly in any closed interval in which f is continuous. At the discontinuity points they converge to the average between left and right limits.

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