2.4 Autoregressive processes: The lag operator

Readers who are acquainted with textbooks on discrete-time stochastic processes should have some knowledge of the backward shift, or lag operator, B (often denoted by L), polynomials in B, such as $a(B) = 1 - a_1 B - a_2 B^2 - \cdots - a_p B^p$, and algebraic manipulation of such polynomials. They should also have encountered statements on the *invertibility* of a polynomial in B, and expressions like $a(B)^{-1} u_t$, which is the stationary solution of the equation $a(B)x_t = u_t$, the latter being the reformulation of equation (2.8). In the present section we review some basic facts on the operator B, polynomials in B as well as more general functions of the operator B. The results obtained in Section 2.3 on the stationary solutions of autoregressive equations will be reobtained here as special cases of more general theorems.

2.4.1 Definition of the lag operator

In Section (1.1) we have defined a stochastic process as a *family* of processes parameterized on the set of integers. As clarified there (see p. 3) "family" is used as a synonym of "function", so that, given the process $x = \{x_t,\ t \in \mathbb{Z}\}$, i.e. the function $t \to x_t$, the family $y = \{x_{t-1},\ t \in \mathbb{Z}\}$, i.e. the function $t \to x_{t-1}$, is a different process.

Definition 2.4 (FIRST DEFINITION OF THE LAG OPERATOR) Let $\mathcal{S}$ be the vector space whose elements are all the stochastic processes defined on the probability space $(\Omega, \mathcal{F}, P)$, i.e. all the families $\{x_t,\ t \in \mathbb{Z}\}$ where x_t is a stochastic variable on $(\Omega, \mathcal{F}, P)$, with the sum of the processes $\{x_t,\ t \in \mathbb{Z}\}$ and $\{y_t,\ t \in \mathbb{Z}\}$, and the product of $\{x_t,\ t \in \mathbb{Z}\}$ by the complex number c, being defined as $\{x_t + y_t,\ t \in \mathbb{Z}\}$ and $\{cx_t,\ t \in \mathbb{Z}\}$ respectively. Then define the map $\mathbf{B} : \mathcal{S} \to \mathcal{S}$ in the following way:

$$\mathbf{B}\,\{x_t,\ t \in \mathbb{Z}\} = \{x_{t-1},\ t \in \mathbb{Z}\}.$$

It should be pointed out that $\mathbf{B}$ maps processes to processes—in the same way as the derivative operator maps functions to functions—not stochastic variables to stochastic variables.

Obviously $\mathbf{B}$ is linear. It is also easily seen that $\mathbf{B}$ is invertible, with $\mathbf{B}^{-1}$ being the linear map $\mathbf{F}$, which is defined in the following way: $\mathbf{F}\,\{x_t,\ t \in \mathbb{Z}\} = \{x_{t+1},\ t \in \mathbb{Z}\}$. As a useful exercise the reader can give the definition of $\mathbf{B}^m$ for any integer $m \neq 0$. Then, denoting by 1 the identity operator on $\mathcal{S}$, define $\mathbf{B}^0 = 1$. The sum $a\mathbf{B}^s + b\mathbf{B}^r$ is obviously defined by $[a\mathbf{B}^s + b\mathbf{B}^r]\{x_t,\ t \in \mathbb{Z}\} = a\mathbf{B}^s\{x_t,\ t \in \mathbb{Z}\} + b\mathbf{B}^r\{x_t,\ t \in \mathbb{Z}\} = \{ax_{t-s} + bx_{t-r},\ t \in \mathbb{Z}\}$, so that, given the polynomial operator $c(\mathbf{B}) = c_0 + c_1\mathbf{B} + \cdots + c_p\mathbf{B}^p$, we have

$$c(\mathbf{B})\{x_t,\ t \in \mathbb{Z}\} = \{c_0 x_t + c_1 x_{t-1} + \cdots + c_p x_{t-p},\ t \in \mathbb{Z}\}.$$

Lastly, the autoregressive equations (2.8) and (2.9) can be rewritten as

$$a(\mathbf{B})\{x_t,\ t \in \mathbb{Z}\} = \{u_t,\ t \in \mathbb{Z}\} \quad \text{and} \quad a(\mathbf{B})\{x_t,\ t \in \mathbb{Z}\} = 0$$

respectively, where $a(\mathbf{B}) = 1 - a_1\mathbf{B} - a_2\mathbf{B}^2 - \cdots - a_p\mathbf{B}^p$.

Now consider for example the case $a(\mathbf{B}) = 1 - b\mathbf{B}$, $b \neq 0$. Independently of the value of b, as we have seen in Section 2.3, there exist infinitely many processes in $\mathcal{S}$, namely $\{Ab^t,\ t \in \mathbb{Z}\}$ where A is any stochastic variable on $(\Omega, \mathcal{F}, P)$, which are mapped to zero by $1-b\mathbf{B}$. Therefore $1-b\mathbf{B}$ is not invertible. On the other hand, by restricting the domain of $\mathbf{B}$ to a subspace of $\mathcal{S}$, as we do in the next definition, $1 - b\mathbf{B}$ becomes invertible for $|b| \neq 1$ and the stationary solution of the equation can be expressed as $(1 - b\mathbf{B})^{-1}\{u_t,\ t \in \mathbb{Z}\}$.

Definition 2.5 (SECOND DEFINITION OF THE LAG OPERATOR) Given the white noise process $\{u_t,\ t \in \mathbb{Z}\}$, let $\mathcal{M}^u$ be the vector subspace of $\mathcal{S}$ whose elements are the moving averages of u_t. Quite obviously $\mathbf{B}$ maps moving averages of u_t to moving averages of u_t, thus $\mathcal{M}^u$ into $\mathcal{M}^u$. Define $B_u : \mathcal{M}^u \to \mathcal{M}^u$ as the restriction of $\mathbf{B}$ to $\mathcal{M}^u$.

In $\mathcal{M}^u$, the autoregressive equation (2.8) can be rewritten as

$$a(B_u)\{x_t,\ t \in \mathbb{Z}\} = \{u_t,\ t \in \mathbb{Z}\}, \tag{2.47}$$

a solution being a member of $\mathcal{M}^u$, i.e. a moving average $y_t = \sum_{k=-\infty}^{\infty} b_k u_{t-k}$, $t \in \mathbb{Z}$. As we have seen in Section 2.3, solving (2.47) is equivalent to finding a square summable solution $\{b_k,\ k \in \mathbb{Z}\}$ of equation (2.40), which is reported here for ease of the reader

$$b_k - a_1 b_{k-1} - \cdots - a_p b_{k-p} = \begin{cases} 0 \text{ for } k \neq 0 \\ 1 \text{ for } k = 0. \end{cases} \tag{2.40}$$

In Section 2.3 we have given the condition for the existence of a solution of (2.40), and shown that the solution, if existent, is unique; see Proposition 2.7. In the present section we show, with a more general and powerful method, that solving (2.40) is equivalent to inverting the map $a(B_u) : \mathcal{M}^u \to \mathcal{M}^u$.

Invertibility of $a(B_u)$ does not contradict the observation above on $1 - b\mathbf{B}$. Indeed, if a moving average $y_t = \sum_{k=-\infty}^{\infty} b_k u_{t-k}$ is a non-trivial solutions of $(1 - bB_u)\{x_t,\ t \in \mathbb{Z}\} = 0$, then

$$b_k - b b_{k-1} = 0.$$

But the solutions of this equation are all of the form $b_k = Cb^k$, which is not square summable unless $C = 0$. More in general, if $a(B_u)\{x_t,\ t \in \mathbb{Z}\} = 0$ had a non-trivial solution in $\mathcal{M}^u$, i.e. a moving average $y_t = \sum_{k=-\infty}^{\infty} b_k u_{t-k}$, then

$$b_k - a_1 b_{k-1} - \cdots - a_p b_{k-p} = 0.$$

But a solutions of this equation cannot be square summable, unless $b_k = 0$ for all k (see Proposition 2.5 (iii)).

Observation 2.8 Of course the argument above proves that (2.40) cannot have more than one square summable solution, i.e. that equation (2.47) cannot have more than one solution in $\mathcal{M}^u$. (This is nothing other than the argument used to prove the uniqueness in proposition 2.7.)

Before we set out to study invertibility of polynomials in B_u, let us further simplify our framework and notation by defining the lag operator in H^u.

Definition 2.6 (THIRD DEFINITION OF THE LAG OPERATOR) Let $x \in H^u$ with representation $x = \sum_{k=-\infty}^{\infty} a_{xk} u_{-k}$. Define the map $\check{B}_u : H^u \to H^u$ as

$$\check{B}_u x = \sum_{k=-\infty}^{\infty} a_{xk} u_{-k-1} = \sum_{k=-\infty}^{\infty} a_{x,k-1} u_{-k}.$$

Because the coefficients a_{xk} are uniquely determined (u_t is an orthogonal sequence), Definition 2.6 is unambiguous. In particular, $\check{B}_u u_t = u_{t-1}$ and, if $y_t = \sum_{k=-\infty}^{\infty} b_k u_{t-k}$,

$$\check{B}_u y_t = \sum_{k=-\infty}^{\infty} b_k u_{t-k-1} = \sum_{k=-\infty}^{\infty} b_k u_{(t-1)-k} = y_{t-1}.$$

As we have seen on p. 49, the map $N : \mathcal{M}^u \to H^u$, associating x_0 with the moving average $\{x_t,\ t \in \mathbb{Z}\}$, is one-to-one and onto, this meaning that each element of H^u is the image of one and only one element of $\mathcal{M}^u$. Moreover,

$$\check{B}_u N\{x_t,\ t \in \mathbb{Z}\} = \check{B}_u x_0 = x_{-1} = N\{x_{t-1},\ t \in Z\} = NL_u\{x_t,\ \in \mathbb{Z}\},$$

for all $\{x_t,\ t \in \mathbb{Z}\} \in M^u$, that is $\check{B}_u = NB_u N^{-1}$ and, more in general,

$$\check{B}_u^s = NB_u^s N^{-1}.$$

As a consequence, if $\{y_t,\ t \in \mathbb{Z}\}$ solves (2.47), then

$$a(\check{B}_u) y_0 = Na(B_u)N^{-1} y_0 = Na(B_u)\{y_t,\ t \in \mathbb{Z}\} = N\{u_t,\ t \in \mathbb{Z}\} = u_0.$$

On the other hand, if $y = \sum_{k=-\infty}^{\infty} a_{yk} u_{-k} \in H^u$ solves

$$a(\check{B}_u)x = u_0, \tag{2.48}$$

then the moving average $y_t = \sum_{k=-\infty}^{\infty} a_{yk} u_{t-k}$, $t \in \mathbb{Z}$, solves (2.47) (the reader is urged to check the details). Thus there is a one to one correspondence between moving averages that solve (2.47) and elements of H^u that solve (2.48).

The correspondence between (2.47) and (2.48) can also be seen in a very simple way. Indeed, both equations are equivalent to equation (2.40). The reason for the long and perhaps a little pedantic argument above is that I want to underline that H^u is a space of stochastic variables, not processes. This having been made clear enough, equation (2.48) can now be seen as one out of an infinity of equations in H^u, one for each t,

$$a(\check{B}_u)x_t = u_t, \tag{2.49}$$

all being equivalent to (2.40). As a consequence, if (2.40) has the unique solution $\{b_k,\ k \in \mathbb{Z}\}$, then the solutions of all equations (2.49) have the 'same coefficients', more precisely, $y_t = \sum_{k=-\infty}^{\infty} b_k u_{t-k}$, the family $\{y_t,\ t \in \mathbb{Z}\}$ being therefore a moving average. As anticipated, this result will be obtained below by inverting $a(\check{B}_u)$, so that the solution of (2.49) will be written as $y_t = a(\check{B}_u)^{-1} u_t$.

In conclusion, problems (2.47) and (2.49) are equivalent. We will concentrate on (2.49) and therefore on H^u and $\check{B}_u$. As no confusion can arise, we will use B instead of $\check{B}_u$, and therefore rewrite (2.49) as

$$a(B)x_t = u_t. \tag{2.50}$$

Observation 2.9 At the risk of being tedious, let me stress that $y_t = Ab^t$, where $A \in H^u$, is *not* a solution of the equation $(1 - bB)x_t = 0$, in H^u. Indeed, using Definition 2.6, $B(Ab^t)$ is *not* equal to Ab^{t-1}. Rather, if $A = \sum_{k=-\infty}^{\infty} c_k u_{-k}$, we have $B(Ab^t) = b^t \sum_{k=-\infty}^{\infty} c_{k-1} u_{-k}$, so that $(1 - bB)y_t = 0$ is equivalent to

$$\sum_{k=-\infty}^{\infty} (c_k - bc_{k-1})u_{-k} = 0,$$

i.e. $c_k - bc_{k-1} = 0$, for which no square summable solution exists.

2.4.2 B as an operator on the Hilbert space H^u. Functions of B

Obviously the map $B : H^u \to H^u$ is linear, that is $B(ax + by) = aBx + bBy$. Moreover, $(Bx) \cdot (By) = x \cdot y$, so that $\|Bx\| = \|x\|$. This implies that if $\lim_{n\to\infty} x_n = x$ then $Bx_n \to Lx$, i.e. B is continuous. A map of a Hilbert space into itself which is linear and continuous is called a *linear operator*, or, briefly, an operator. Thus B deserves the label 'lag operator'. Because $(Bx)\cdot(By) = x\cdot y$, B is called a *unitary* operator.

If A_1 and A_2 are operators on H^u, their product A_1A_2, defined by $(A_1A_2)x = A_1(A_2x)$, is linear and continuous, thus an operator. The definition of A^n, with n a positive integer, is obtained recursively from $A^1 = A$ and $A^n = AA^{n-1}$. Denoting by 1 the identity operator, which maps x to x for all $x \in H^u$, we set $A^0 = 1$. Given A, if there exists an operator C such that $AC = CA = 1$, then A is invertible. If $AC = CA = 1$ and $AC_1 = C_1A = 1$, then $C = 1\,C = (C_1A)C = C_1(AC) = C$ (the associative property for the product of operators is obvious). In other words, if an inverse exists it is unique. We denote it by A^{-1}. (On the basic notions about the algebra of operators on Hilbert spaces see e.g. [8], Sections 19 and 20.)

Observation 2.10 The reader acquainted with linear algebra on finite-dimensional vector spaces may be surprised by the definition of an operator as a linear *and* continuous map. Obviously, a linear map whose domain is a finite-dimensional vector space is continuous, but this is not necessarily the case when the space is infinite dimensional. An example of a linear non-continuous map can be found in [10], pp. 24 and 204.

Observation 2.11 If a linear operator A on a finite-dimensional vector space is one-to-one, that is if $Ax = Ay$ implies $x = y$, then A is onto and is invertible, the inverse being linear and continuous. This is not true for infinite-dimensional spaces. Consider for example $A : H^u \to H^u$ defined as

$$Ax = A\sum_{k=-\infty}^{\infty} a_{xk}u_{-k} = \sum_{k=-\infty}^{\infty} a_{xk}u_{-2k}.$$

In this case A is one-to-one but is not onto and therefore is not invertible. However, if A is linear, continuous, one-to-one and onto, so that a set-theoretic inverse exists, then the inverse is continuous; see [10], pp. 27 and 205.

Trivially B is one-to-one and onto, and the map F defined as

$$Fx = \sum_{k=-\infty}^{\infty} a_{xk}u_{-k+1} = \sum_{k=-\infty}^{\infty} a_{x,k+1}u_{-k},$$

is continuous, thus an operator, and has the property that $FBx = BFx = x$ for all $x \in H^u$, so that $F = B^{-1}$ and $B = F^{-1}$.

An important family of maps is obtained by taking "functions of B". Consider the subset of H^u whose members are *finite* linear combinations of the u's, i.e. all vectors x such that, for some integer m, $a_{xk} = 0$ for $|k| > m$. We denote this subset by $\check{H}^u$. $\check{H}^u$ is a vector subspace of H^u, which is not closed but is dense in H^u.

Exercise 2.10 Prove that every vector of H^u is the limit of a sequence of vectors belonging to $\check{H}^u$ (quite obvious), thus $\check{H}^u$ is dense in H^u. But, trivially, $\check{H}^u$ does not coincide with H^u, so that the statement in the text is proved. Incidentally, the space H^u is isomorphic to $l^2(-\infty, \infty)$, see (2.5), with the definition of the map Ψ, while the subspace $\check{H}^u$ is isomorphic to the subspace ℓ^2, defined in Exercise 1.11.

Now, given any square summable sequence $b = \{b_k,\ k \in \mathbb{Z}\}$ define the map $b(B) : \check{H}^u \to H^u$ in the following way:

$$b(B)x = \lim_{s\to\infty} \sum_{k=-s}^{s} b_k(B^k x) = \sum_{k=-\infty}^{\infty} b_k(B^k x).$$

This definition obviously makes sense if $x = u_h$:

$$b(B)u_h = \sum_{k=-\infty}^{\infty} b_k u_{h-k},$$

convergence being a consequence of square summability of the sequence b. In general, if $x = \sum_{h=-m}^{m} a_{xh} u_{-h}$, it is very easy to see that

$$b(B)x = \sum_{h=-m}^{m} a_{xh} \sum_{k=-\infty}^{\infty} b_k u_{-h-k}.$$

Obviously $b(B)$ is linear. We denote by $\mathcal{A}(B)$ the set of maps just defined.

Observation 2.12 It is very important to remark that if $\{y_t,\ t \in \mathbb{Z}\}$ is a moving average of u_t, $y_t \in \check{H}^u$, and $b(B) \in \mathcal{A}(B)$, then $\{b(B)y_t, t \in \mathbb{Z}\}$ is a moving average of u_t. Conversely, if $A : \check{H}^u \to H^u$ is linear and $\{Au_t,\ t \in \mathbb{Z}\}$ is a moving average of u_t, then $A \in \mathcal{A}(B)$. The details are left to the reader.

Observation 2.13 The maps of $\mathcal{A}(B)$ can be used to simplify notation. Instead of $x = \sum_{k=-\infty}^{\infty} a_{xk} u_{-k}$ and $x_t = \sum_{k=-\infty}^{\infty} a_k u_{t-k}$ we can write

$$x = a_x(B)u_0, \qquad x_t = a(B)u_t,$$

respectively, the meaning of $a_x(B)$ and $a(B)$ being obvious.

Now define $\mathcal{B}(B)$ as the subset of $\mathcal{A}(B)$ containing all *bounded* maps, i.e. the maps $b(B)$ such that

$$\sup_{x \in \check{H}^u,\ ||x||=1} ||b(B)x|| = C_b < \infty.$$

It is very easy to prove that if $b(B)$ is bounded then

$$||b(B)x|| \leq C_b ||x|| \tag{2.51}$$

for all $x \in \check{H}^u$.

Now let $b(B) \in \mathcal{B}(B)$ and $x \in H^u$, with $x = \sum_{k=-\infty}^{\infty} a_{xk} u_{-k}$. Obviously,

$$x = \lim_{m \to \infty} \sum_{k=-m}^{m} a_{xk} u_{-k} = \lim_{m \to \infty} x_m,$$

x_m denoting the m-th partial sum. Using (2.51),

$$||b(B)x_m - b(B)x_n|| = ||b(B)(x_m - x_n)|| \leq C_b ||x_m - x_n||,$$

so that convergence of x_m implies that $b(B)x_m$ is a Cauchy sequence and therefore convergent. We define, for $x \in H^u$,

$$b(B)x = \lim_{m \to \infty} b(B)x_m,$$

in this way extending the domain of the map $b(B)$ to the whole H^u. The reader will easily see that $b(B)$, so extended, is linear, that

$$\sup_{x \in H^u,\ ||x||=1} ||b(B)x|| = C_b,$$

and that (2.51) holds for all $x \in H^u$. A linear map A on a Hilbert space that is bounded, i.e. such that $\sup ||Ax|| = C_A < \infty$ for $||x|| = 1$, is continuous and therefore an operator. For, if $y_n \to y$ then

$$||Ay_n - Ay|| \leq C_A ||y_n - y|| \to 0.$$

In conclusion, the maps belonging to $\mathcal{B}(B)$ can be extended to operators on H^u. With a slight inaccuracy we denote by $b(B)$ both the map with domain on $\check{H}^u$ and its extension, and say that $\mathcal{B}(B)$ is a set of operators on H^u.

Example 2.7 A LINEAR MAP NOT BELONGING TO $\mathcal{A}(B)$. The map which linearly extends $u_k \to u_{-k}$, i.e.

$$x = \sum_{k=-\infty}^{\infty} a_{xk} u_{t-k} \to \sum_{k=-\infty}^{\infty} a_{xk} u_{-t+k},$$

is an operator on H^u (quite obvious) but does not belong to $\mathcal{A}(B)$. Indeed, the process $\{u_t,\ t \in \mathbb{Z}\}$ is not mapped on a moving average of u_t (see Observation 2.12).

Example 2.8 MAPS BELONGING TO $\mathcal{A}(B)$ BUT NOT TO $\mathcal{B}(B)$. Assume that b_k is real and non-negative, that $b_k = 0$ for $k < 0$, that $\sum_{k=0}^{\infty} b_k = +\infty$ and that $\sum_{k=0}^{\infty} |b_k|^2 < \infty$. For example, take $b_k = \dfrac{1}{k+1}$ for $k \geq 0$ (see Exercise 1.5). Setting

$$x_n = \frac{1}{\sigma_u \sqrt{n}}(u_0 + u_{-1} + \cdots u_{-n+1}),$$

so that $||x_n|| = 1$, we have:

$$b(B)x_n = \frac{1}{\sigma_u \sqrt{n}}[b_0 u_0 + (b_0 + b_1)u_{-1} + \cdots + (b_0 + b_1 + \cdots + b_{n-1})u_{-n+1}] + \cdots$$

Thus

$$\begin{aligned} ||b(B)x_n||^2 &\geq \frac{1}{n}[b_0^2 + (1 + b_1)^2 + \cdots + (b_0 + b_1 + \cdots + b_{-n+1})^2] \\ &= \frac{1}{n}(c_0 + c_1 + \cdots + c_{n-1}), \end{aligned}$$

where $c_h = \sum_{s=0}^{h-1} b_s^2$. Because c_h is a non decreasing diverging sequence, given $M > 0$ there exists an integer n_M such that if $h > n_M$ then $c_h > M + 1$. For $n > n_M$,

$$\begin{aligned} \frac{1}{n}(\sum_{j=0}^{n-1} c_j) &= \frac{1}{n}(\sum_{j=0}^{n_M} c_j) + \frac{n - n_M - 2}{n}\ \frac{(c_{n_M+1} + \cdots + c_{n-1})}{n - n_M - 2} \\ &\geq \frac{1}{n}\sum_{j=0}^{n_M} c_j + \frac{n - n_M - 2}{n}(M + 1) \end{aligned}$$

Thus, given $M > 0$, we can find n such that $||b(B)x_n||^2 > M$, this implying that $b(B)$ is not bounded.

Example 2.9 ABSOLUTELY CONVERGENT SEQUENCES. Assume that the sequence $\{b_k,\ k \in \mathbb{Z}\}$ is absolutely convergent, i.e. that

$$\sum_{k=-\infty}^{\infty} |b_k| < \infty.$$

This implies that $\sum b_k$ and $\sum |b_k|^2$ are both convergent. By definition, if $x \in \check{H}^u$,

$$b(B)x = \lim_{m\to\infty} \sum_{k=-m}^{m} b_k(B^k x).$$

Assuming that $||x|| = 1$, by the triangular inequality, and because B is unitary,

$$\left|\left|\sum_{k=-m}^{m} b_k(B^k x)\right|\right| \leq \sum_{k=-m}^{m} |b_k|.$$

Thus $||b(B)x|| \leq \sum_{k=-\infty}^{\infty} |b_k| < \infty$, so that $b(B)$ is bounded. In Section 2.4.3 we show that absolute convergence of the sequence b_k, though sufficient, is not necessary for $b(B) \in \mathcal{B}(L)$. Of course absolutely convergent operators include finite sums

$$\sum_{k=-m}^{m} b_k B^k = a_{-m}B^{-m} + \cdots + a_0 + \cdots a_m B^m = a_{-m}F^m + \cdots + a_0 + \cdots a_m B^m$$

and, in particular, polynomials in B or F.

Let us now list some basic facts on the maps of $\mathcal{A}(B)$ and $\mathcal{B}(B)$, further analysis being postponed to Section 2.4.3.

FACT 1. If $a(B)$ and $b(B)$ belong to $\mathcal{A}(B)$ and correspond, respectively, to the sequences $\{a_k,\ k \in \mathbb{Z}\}$ and $\{b_k,\ k \in \mathbb{Z}\}$, the map $a(B) + b(B)$ belongs to $\mathcal{A}(B)$ and corresponds to the sequence $\{a_k + b_k,\ k \in \mathbb{Z}\}$. If $a(B)$ and $b(B)$ belong to $\mathcal{B}(B)$, then $a(B) + b(B)$ belongs to $\mathcal{B}(B)$ (from the definition of boundedness and the triangular inequality).

FACT 2. If $a = \{a_k,\ k \in \mathbb{Z}\}$ and $b = \{b_k,\ k \in \mathbb{Z}\}$ belong to $l^2(-\infty, \infty)$ the *convolution* of a and b, denoted by $a * b$, is, by definition, the sequence $c = \{c_k,\ k \in \mathbb{Z}\}$, with

$$c_h = \sum_{k=-\infty}^{\infty} a_k b_{h-k}.$$

Note that c_h is the inner product of the two square summable sequences

$$\begin{matrix} \cdots & a_{-1} & a_0 & a_1 & \cdots \\ \cdots & \bar{b}_{h+1} & \bar{b}_h & \bar{b}_{h-1} & \cdots \end{matrix}$$

and is therefore finite. However, the sequence $c = \{c_k,\ k \in \mathbb{Z}\}$ is not necessarily square summable (examples in Section 2.4.3). The convolution operation is commutative: $a * b = b * a$.

Now, if $a(B) \in \mathcal{A}(B)$ and

$$b(B) = \sum_{k=-m}^{m} b_k B^k,$$

then the map $b(L)a(B) : \check{H}^u \to H^u$ ($b(B)$ can be extended to the whole H^u) belongs to $\mathcal{A}(B)$. Setting $c(B) = b(B)a(B)$, the sequence corresponding to $c(B)$ is the convolution of b and a. Moreover, because $b(B)$ maps $\check{H}^u$ into $\check{H}^u$, the map $a(B)b(B)$ makes sense and is equal to $b(B)a(B)$. If $a(B)b(B)u_0 = u_0$, then $a(B) = (b(B))^{-1}$. This is left to the reader as an exercise.

FACT 3. If $a(B)$ and $b(B)$ belong to $\mathcal{B}(B)$, then $a(B)b(B)$ belong to $\mathcal{B}(B)$. Setting $c(B) = b(B)a(B)$, the sequence corresponding to $c(B)$ is the convolution of b and a. Moreover, $b(B)a(B) = a(B)b(B)$. This will be proved in Section 2.4.3.

Sometimes we may find useful the following representation:

$$b(B) = \sum_{k=-\infty}^{\infty} b_k B^k.$$

Since we have not defined convergence for operators on H^u, the right hand side can only be justified as an heuristically convenient expression. In particular, the product of $a(B)$ and $b(B)$ can be thought of as the multiplication of 'infinite polynomials', the operation consisting in taking all possible products and regrouping according to the exponent of B, just like in the elementary finite-polynomial case. This is precisely what convolution does. It should be kept in mind however that infinite polynomial multiplication can be carried out within the limits established in Facts 2 and 3 (further clarification in Section 2.4.3).

Exercise 2.11 Prove that $1-bB$, with $|b| \neq 1$, is invertible and that $(1-bB)^{-1} = 1 + bB + b^2B^2 + \cdots$, for $|b| < 1$, and $(1 - bB)^{-1} = b^{-1}F + b^{-2}F^2 + \cdots$, for $|b| > 1$. Use Fact 2 and compare with Proposition 2.4.

In Section 2.4.3 we characterize the operators of $\mathcal{B}(B)$ and invertible operators of $\mathcal{B}(B)$. Moreover, we discuss the following generalization of the autoregressive equation:

$$a(B)x_t = b(B)u_t, \tag{2.52}$$

where $a(B)$ and $b(B)$ belong to $\mathcal{A}(B)$. To do so we will exploit an isomorphism between H^u and $L^2([-\pi, \pi])$.

2.4.3 From H^u to $L^2([-\pi, \pi])$

Consider the space $L^2([-\pi\ \pi])$ (see Definition 1.4) and the function $\mathbf{g} \in L^2([-\pi\ \pi])$ defined as

$$\mathbf{g}(\theta) = e^{-i\theta}.$$

Defining $\mathbf{g}^k$ by $\mathbf{g}^k(\theta) = [\mathbf{g}(\theta)]^k = e^{-ik\theta}$, we have:

$$\mathbf{g}^k\,\mathbf{g}^h = \mathbf{g}^{k+h}, \quad (\mathbf{g}^k)^h = \mathbf{g}^{kh}.$$

Moreover, for the inner product between $\mathbf{g}^k$ and $\mathbf{g}^h$ we have

$$\begin{aligned}
\mathbf{g}^k \cdot \mathbf{g}^h &= \int_{-\pi}^{\pi} e^{-ik\theta}\overline{e^{-ih\theta}}d\theta = \int_{-\pi}^{\pi} e^{-ik\theta}e^{ih\theta}d\theta \\
&= \int_{-\pi}^{\pi} e^{-i(k-h)}\theta d\theta \\
&= \int_{-\pi}^{\pi} \cos[(k-h)\theta]d\theta - i\int_{-\pi}^{\pi} \sin[(k-h)\theta]d\theta \\
&= \begin{cases} 0 & \text{if } k \neq h \\ 2\pi & \text{if } k = h \end{cases}
\end{aligned}$$

Thus $\{\mathbf{g}^k,\ k \in \mathbb{Z}\}$ is an orthogonal family whose members have all the same modulus $\sqrt{2\pi}$. Now consider $H^e = \overline{\text{sp}}(\mathbf{g}^k,\ k \in \mathbb{Z})$. Orthogonality of the functions $\mathbf{g}^k$ implies that if $f \in H^e$ then

$$f = \sum_{k=-\infty}^{\infty} a_{fk}\mathbf{g}^k \tag{2.53}$$

(mean square convergence), where

$$a_{fk} = \frac{f \cdot \mathbf{g}^k}{\|\mathbf{g}^k\|^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{ik\theta}d\theta. \tag{2.54}$$

The Hilbert space H^e is 'naturally' related to the space H^u by the map

$$\Phi : H^u \to H^e,$$

defined in the following way. If $x = \sum_{k=-\infty}^{\infty} a_{xk} u_{-k}$, then

$$\Phi(x) = \Phi\left(\sum_{k=-\infty}^{\infty} a_{xk} u_{-k}\right) = \sqrt{\frac{\sigma_u^2}{2\pi}} \sum_{k=-\infty}^{\infty} a_{xk} \mathbf{g}^k.$$

In particular, $\Phi(u_t) = \sqrt{\frac{\sigma_u^2}{2\pi}} \mathbf{g}^{-t}$. The map Φ is an isomorphism between H^u and H^e, i.e. it is one-to-one, onto, linear, preserves the inner product and therefore the norm. This is an obvious consequence of Proposition 1.4.

Observation 2.14 I prefer using the symbols $\mathbf{g}^k$ instead of the alternative $e^{-ik(\cdot)}$. The latter are obviously more direct but produce awkward formulas.

Up until now this construction does not seem particularly interesting. Any other set of orthogonal functions would produce a subspace in $L^2([-\pi\ \pi])$, which would be isomorphic to H^u. However, there are two facts that make $\{\mathbf{g}^k,\ k \in \mathbb{Z}\}$ special.

Firstly, $H^e = L^2([-\pi\ \pi])$. Precisely, let $f \in L^2([-\pi\ \pi])$ and let

$$f = \sum_{k=-\infty}^{\infty} a_{fk} \mathbf{g}^k + R_f, \tag{2.55}$$

with $R_f \perp g_k,\ k \in \mathbb{Z}$, be the orthogonal decomposition of f with respect to H^e. A basic result in Fourier theory is that $R_f = 0$ for all $f \in L^2([-\pi\ \pi])$, this being a consequence of the following statement:

Proposition 2.9 Let g be integrable in $[-\pi\ \pi]$. If $\int_{-\pi}^{\pi} g(\theta) e^{ik\theta} d\theta = 0$ for all integers k then $g(\theta) = 0$ a.e. in $[-\pi\ \pi]$.

This result will be commented upon in Section 2.5. A consequence of it is that expansion (2.53) establishes an isomorphism between H^u and the whole $L^2([-\pi\ \pi])$. Moreover, it establishes an isomorphism between $L^2([-\pi\ \pi])$ and $l^2(-\infty, \infty)$. Precisely, given $f \in L^2([-\pi\ \pi])$, define

$$\Xi : L^2([-\pi\ \pi]) \to l^2(-\infty, \infty),$$

as the map

$$\Xi(f) = \{a_{fk}\sqrt{2\pi},\ k \in \mathbb{Z}\}$$

(an obvious consequence of Proposition 1.5). The reader will easily see that

$$\Phi \Xi = \Psi,$$

where Ψ has been defined in (2.5), p. 47.

Observation 2.15 The choice of $L^2([-\pi\ \pi])$ is only a convenience. If we take $L^2([a\ b])$ instead, then the functions $\mathbf{g}^k$ should be replaced by the functions

$$\mathbf{h}^k : \theta \to e^{-i\frac{2\pi k}{b-a}\theta},$$

$k \in \mathbb{Z}$, the coefficients a_{fk} would be determined as

$$a_{fk} = \frac{1}{b-a}\int_a^b f(\theta)\mathbf{h}^k(\theta)d\theta,$$

etc. The reader is recommended to check that all the developments below still hold.

Secondly, consider an operator A on H^u. Given the isomorphism Φ the counterpart of A on $L^2([-\pi\ \pi])$, call it $\hat{A}$, is defined as

$$\hat{A}f = \Phi(A(\Phi^{-1}(f))),$$

and, viceversa, given an operator on $L^2([-\pi\ \pi])$, its counterpart on H^u is defined using Φ first and then Φ^{-1}. Consider in particular the operator corresponding to B, call it $\hat{B}$.

$$\begin{aligned}\hat{B}(f) &= \Phi(L(\Phi^{-1}(\textstyle\sum_{k=-\infty}^{\infty} a_{fk}\mathbf{g}^k))) \\ &= \Phi(\textstyle\sum_{k=-\infty}^{\infty} a_{f,k-1}u_{-k}) = \sum_{k=-\infty}^{\infty} a_{f,k-1}\mathbf{g}^k = \mathbf{g}f.\end{aligned}$$

Thus the operator $\hat{B}$, corresponding to B, is the *multiplication operator* which maps f into $\mathbf{g}f$. Moreover, denoting by $\check{H}^e$ the set of all functions of $L^2([-\pi\ \pi])$ that are finite sums

$$\sum_{k=-m}^{m} a_k\mathbf{g}^k,$$

which obviously corresponds to $\check{H}^u$, the counterpart of the map $b(B) \in \mathcal{A}(B)$, with $b(B) = \sum_{h=-\infty}^{\infty} b_k B^k$, is the multiplication map:

$$f \to \left[\sum_{h=-\infty}^{\infty} b_k\mathbf{g}^k\right] f.$$

Conversely, given any function $g \in L^2([-\pi\ \pi])$, the multiplication map

$$f \to g\ f,$$

whose domain and range are $\check{H}^e$ and $L^2([-\pi\ \pi])$ respectively, is the counterpart of the map $\sum_{k=-\infty}^{\infty} a_{gk}B^k$, which belongs to $\mathcal{A}(B)$. In conclusion, the map Φ

establishes a one-to-one correspondence between the maps of $\mathcal{A}(B)$ and the multiplication maps on $\check{H}^e$. Given the map $b(B)$ we denote by $b(\mathbf{g})$ the function $\sum_{k=-\infty}^{\infty} b_k \mathbf{g}^k$. If $x \in \check{H}^u$,

$$\Phi(b(B)x) = b(\mathbf{g})[\Phi(x)].$$

Now we are almost ready for a very elegant characterization of the operators of $\mathcal{B}(B)$.

Definition 2.7 Let $f \in L^2([-\pi\ \pi])$. The function f is *essentially bounded* if there exists a subset A of $[-\pi\ \pi]$ whose measure is 2π and such that f is bounded on A.

Exercise 2.12 Show that if $g \neq f$ only on a zero measure set and f is essentially bounded then g is essentially bounded. We recall that in $L^2([-\pi\ \pi])$ functions that differ only on a zero-measure set are equivalent (see p. 23). Thus all functions that are equivalent to an essentially bounded function are essentially bounded.

Exercise 2.13 Let

$$\begin{aligned} f(\theta) &= \begin{cases} n \text{ for } \theta = 1/n, n \in \mathbb{Z}, n \neq 0 \\ 1 \text{ otherwise,} \end{cases} \\ g(\theta) &= \theta^{\alpha}, \end{aligned}$$

with $-.5 < \alpha < 0$. Show that f is essentially bounded, though not bounded, whereas g is not essentially bounded.

Proposition 2.10 Let $b(B) \in \mathcal{A}(B)$. Then $b(B) \in \mathcal{B}(B)$ if and only if $b(\mathbf{g})$ is essentially bounded.

PROOF. If $b(\mathbf{g})$ is essentially bounded, then, setting $g = b(\mathbf{g})$, for a positive finite M we have $|g(\theta)| \leq M$, for θ a.e. in $[-\pi\ \pi]$. Therefore, for all $f \in \check{H}^e$ (but also for all $f \in L^2([-\pi\ \pi])$),

$$||b(\mathbf{g})f||^2 = \int_{-\pi}^{\pi} |g(\theta)f(\theta)|^2 d\theta \leq M^2 \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta = M||f||^2.$$

Thus $b(\mathbf{g})$, and therefore $b(B)$, is bounded.

To prove the converse, observe firstly that if $b(B) \in \mathcal{B}(B)$ then the multiplication map $b(\mathbf{g}) : \check{H}^u \to L^2([-\pi\ \pi])$ can be extended to the whole $L^2([-\pi\ \pi])$ and that such extension is the multiplication map $f \to b(\mathbf{g})f$, for all $f \in L^2([-\pi\ \pi])$.

Thus $b(B) \in \mathcal{B}(B)$ implies that $b(\mathbf{g})\, f \in L^2([-\pi\ \pi])$ for all $f \in L^2([-\pi\ \pi])$. Assume now that $g = b(\mathbf{g})$ is not essentially bounded. For $k \geq 0$ define

$$A_k = \{\theta,\ k \leq |g(\theta)| < k+1\},$$

and

$$r_k = \begin{cases} \frac{1}{\sqrt{\mu(A_k)}} \text{ if } \mu(A_k) > 0 \\ 0 \text{ if } \mu(A_k) = 0. \end{cases}$$

Then define

$$f(\theta) = \begin{cases} 0 \text{ if } \theta \in A_0 \\ \frac{r_k}{k} \text{ if } k > 0 \text{ and } \theta \in A_k. \end{cases}$$

We have

$$\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta = \sum_{k=0}^{\infty} \int_{A_k} |f(\theta)|^2 d\theta \leq 0 + 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots < \infty,$$

so that $f \in L^2([-\pi\ \pi])$. Now let us consider the function gf. We have

$$\int_{-\pi}^{\pi} |g(\theta)f(\theta)|^2 d\theta = \sum_{k=0}^{\infty} \int_{A_k} |g(\theta)|^2 |f(\theta)|^2 d\theta \geq s_0 + s_1 + \cdots,$$

where $s_0 = 0$ and, for $k > 0$,

$$s_k = \begin{cases} 0 \text{ if } \mu(A_k) = 0 \\ 1 \text{ if } \mu(A_k) > 0. \end{cases}$$

The assumption that g is not essentially bounded implies that there is an infinite number of integers k for which $s_k = 1$. Thus gf does not belong to $L^2([-\pi\ \pi])$. This completes the proof. (A much more elegant and quick proof can be found in [10], Problem 50, pp. 31 and 212; take the demonstration above as an exercise.)

Observation 2.16 We say that $g \in L^2([-\pi\ \pi])$ is continuous if it differs from a continuous function only on a zero-measure set. If f and g are continuous and differ only on a zero measure set then they are equal everywhere (easy to prove). Now, if $\{b_k,\ k \in \mathbb{Z}\}$ is absolutely convergent then the function $\sum_{k=-\infty}^{\infty} b_k \mathbf{g}^k$ is continuous. For, the convergence of $\sum_{k=-m}^{m} b_k \mathbf{g}^k$ is uniform in $[-\pi\ \pi]$ and each of the partial sums is continuous. On the other hand, essentially bounded functions are not continuous in general: for example, as the reader may check, the function

$$h(\theta) = \begin{cases} 1 \text{ if } |\theta| < 1 \\ 0 \text{ if } |\theta| \geq 1 \end{cases}$$

is not continuous, nor equivalent to any continuous function. Thus the Fourier coefficients of h are not absolutely convergent and we have proved that absolute convergence is not necessary for $b(B) \in \mathcal{B}(B)$ (see Example 2.9).

We can now characterize those operators of $\mathcal{B}(B)$ that are invertible.

Definition 2.8 Let $g \in L^2([-\pi\ \pi])$. g is essentially bounded away from zero if there exists $\eta > 0$ such that $g(\theta) > \eta$ for θ a.e. in $[-\pi\ \pi]$.

If g is essentially bounded and bounded away from zero, then $1/g$ is essentially bounded away from zero and bounded. This is what we need to prove the following statement.

Proposition 2.11 Let $b(B) \in \mathcal{B}(B)$. $b(B)$ is invertible if and only if $b(\mathbf{g})$ is essentially bounded away from zero. In that case, let $1/b(\mathbf{g}) = \sum_{k=-\infty}^{\infty} d_k \mathbf{g}^k$ be the Fourier expansion of $1/b(\mathbf{g})$. Then

$$[b(B)]^{-1} = \sum_{k=-\infty}^{\infty} d_k B^k.$$

Observation 2.17 Assuming that g is essentially bounded, the existence of a function h in $L^2([-\pi\ \pi])$ such that $gh = 1$ a.e. in $[-\pi\ \pi]$ does not imply that the multiplication operator associated with g has an inverse. For example, let $g(\theta) = \theta^{\alpha}$, with $0 < \alpha < .5$. In this case $h(\theta) = \theta^{-\alpha}$ belongs to $L^2([-\pi\ \pi])$ and $gh = 1$ a.e. in $[-\pi\ \pi]$. However as g is not essentially bounded away from zero, h is not essentially bounded and therefore does not define a multiplication operator.

The following statement implies Fact 3, Section 2.4.2.

Proposition 2.12 Assume that f, g and the product $f\,g$ belong to $L^2([-\pi\ \pi])$. Then $a_{fg} = a_f * a_g$. Conversely, if $a = \{a_k,\ k \in \mathbb{Z}\}$, $b = \{a_k,\ k \in \mathbb{Z}\}$ and the convolution $a * b$ belong to $l^2(-\infty, \infty)$, then $(a * b)(\mathbf{g}) = a(\mathbf{g})\, b(\mathbf{g})$.

PROOF. Firstly, note that, using (1.21),

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)g(\theta)e^{is\theta}d\theta = \frac{1}{2\pi} f\cdot[\bar{g}\mathbf{g}^s] = \sum_{n=-\infty}^{\infty} a_{fn}a_{g,s-n}, \tag{2.56}$$

i.e. the convolution $a_f * a_g$ is the sequence of integrals in (2.56), this result holding irrespectively of whether fg belongs to $L^2([-\pi\ \pi])$ or not. If $fg \in L^2([-\pi\ \pi])$ then (2.56) is $a_{fg,s}$, so that the convolution $a_f * a_g$ belongs to $l^2(-\infty,\infty)$. Conversely, let $c = a*b$. Because $c \in l^2(-\infty,\infty)$, $H = \sum_{k=-\infty}^{\infty} c_k \mathbf{g}^k \in L^2([-\pi\ \pi])$. Applying (2.56) to $a(\mathbf{g})b(\mathbf{g})$:

$$[H - a(\mathbf{g})b(\mathbf{g})]\cdot \mathbf{g}^k = 0$$

for all k, so that, by Proposition 2.9, $H = a(\mathbf{g})b(\mathbf{g})$.

Example 2.10 Examples of sequences of $l^2(-\infty,\infty)$ whose convolution does not belong to $l^2(-\infty,\infty)$ are obtained by taking the Fourier coefficients of functions of $L^2([-\pi\ \pi])$ whose product does not belong to $L^2([-\pi\ \pi])$. For example, $f(\theta) = \theta^{-.3}$, $h(\theta) = \theta^{-.4}$. The function $f(\theta)h(\theta) = \theta^{-.7}$, whose squared modulus is $|\theta|^{-1.4}$, is not in $L^2([-\pi\ \pi])$.

Lastly we return to the main task of the section, that is equation (2.52), rewritten below:

$$a(B)x_t = b(B)u_t, \tag{2.52}$$

and its counterpart in $L^2([-\pi\ \pi])$:

$$a(\mathbf{g})f_t = b(\mathbf{g})\mathbf{g}^{-t}. \tag{2.57}$$

Proposition 2.13 Assume that $a(B)$ and $b(B)$ belong to $\mathcal{A}(B)$ and that $a(\mathbf{g}) \neq 0$ a.e. in $[-\pi\ \pi]$, so that the function $1/a(\mathbf{g})$ is Lebesgue measurable and finite a.e. in $[-\pi\ \pi]$. If $b(\mathbf{g})/a(\mathbf{g}) \in L^2([-\pi\ \pi])$ then (2.52) has the unique solution $y_t = d(B)u_t$, where $\sum_{k=-\infty}^{\infty} d_k\mathbf{g}^k$ is the Fourier expansion of $b(\mathbf{g})/a(\mathbf{g})$, that is, setting $c = b(\mathbf{g})/a(\mathbf{g})$,

$$d_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} c(\theta)e^{ik\theta}d\theta.$$

Of course $\{y_t,\ t\in\mathbb{Z}\}$ is a moving average. Conversely, if there exists $\tau\in\mathbb{Z}$ such that the equation $a(B)x_\tau = b(B)u_\tau$ has a solution $y_\tau \in H^u$, then $b(\mathbf{g})/a(\mathbf{g}) \in L^2([-\pi\ \pi])$.

PROOF. If $b(\mathbf{g})/a(\mathbf{g}) \in L^2([-\pi\ \pi])$ then $f_t = [b(\mathbf{g})/a(\mathbf{g})]\mathbf{g}^{-t}$ is a solution of (2.57) belonging to $L^2([-\pi\ \pi])$. For,

$$|[\mathrm{b}(\mathbf{g})/\mathrm{a}(\mathbf{g})]\mathbf{g}^{-\tau}|^2 = |\mathrm{b}(\mathbf{g})/\mathrm{a}(\mathbf{g})|^2. \tag{2.58}$$

Moreover, if $h_t \in L^2([-\pi\ \pi])$ and solves (2.57), then $h_t = [b(\mathbf{g})/a(\mathbf{g})]\mathbf{g}^{-t}$. If $f_\tau \in L^2([-\pi\ \pi])$ and solves $a(\mathbf{g})x_\tau = b(\mathbf{g})\mathbf{g}^{-\tau}$, then $[b(\mathbf{g})/a(\mathbf{g})]\mathbf{g}^{-\tau} \in L^2([-\pi\ \pi])$. Therefore, by (2.58), $b(\mathbf{g})/a(\mathbf{g}) \in L^2([-\pi\ \pi])$.

Exercise 2.14 If the function $a(\mathbf{g})$ vanishes on a set of positive measure the solution of (2.52), if existing, is not unique. This is left to the reader.

We want to analyze in detail two cases of equation (2.52): (1) *Autoregressive-moving average* processes (ARMA), in which the maps $a(B)$ and $b(B)$ are polynomials:

$$a(B) = 1 - a_1 B - \cdots - a_p B^p, \quad b(B) = 1 + b_1 B + \cdots + b_q B^q,$$

and (2) Processes resulting from *fractional differencing*, in which $b(B) = 1$ and

$$a(B) = (1 - B)^d,$$

where $0 < d < 1/2$.

Two preliminary results are needed.

Proposition 2.14 Suppose that $f_n \to f$ in $L^2([-\pi\ \pi])$. Then $\{a_{f_n k},\ k \in \mathbb{Z}\} \to \{a_{fk},\ k \in \mathbb{Z}\}$ in $l^2(-\infty, \infty)$. Conversely, if $\{a_{nk},\ k \in \mathbb{Z}\} \to \{a_k,\ k \in \mathbb{Z}\}$ in $l^2(-\infty, \infty)$, then $\sum_{k=-\infty}^{\infty} a_{nk}\mathbf{g}^k \to \sum_{k=-\infty}^{\infty} a_k \mathbf{g}^k$ in $L^2([-\pi\ \pi])$.

The proof, hardly needed, is left to the reader.

Proposition 2.15 Let $f : \mathbb{C} \to \mathbb{C}$ be analytic in an open connected set containing the disk of radius $R > 1$ and center at zero. In this case, (see [1], pp. 179-84), f can be expanded in Taylor series

$$f(z) = f(0) + \frac{1}{1!}f'(0)z + \frac{1}{2!}f''(0)z^2 + \cdots,$$

the series converging uniformly for $|z| \le R' < R$. Then, setting $f_1(\theta) = f(e^{-i\theta})$ and $f_2(\theta) = f(e^{i\theta})$, the Fourier expansion of f_1 and f_2 are

$$f_1 = f(0) + \frac{1}{1!}f'(0)\mathbf{g} + \frac{1}{2!}f''(0)\mathbf{g}^2 + \cdots, \tag{2.59}$$

which is one-sided in the positive powers of $\mathbf{g}$, and

$$f_2 = f(0) + \frac{1}{1!} f'(0)\mathbf{g}^{-1} + \frac{1}{2!} f''(0)\mathbf{g}^{-2} + \cdots , \tag{2.60}$$

which is one-sided in the negative powers of $\mathbf{g}$.

PROOF. As $R > 1$ we can take $R' > 1$, so that the Taylor series converges uniformly for $|z| = 1$. As a consequence, both (2.59) and (2.60) converge uniformly for $\theta \in [-\pi\ \pi]$, and therefore converge in mean square (see Exercise 1.17). But then, by Proposition 1.4, (3), (2.59) and (2.60) are the Fourier expansions of f_1 and f_2 respectively.

CASE 1. Consider the rational function

$$c(z) = \frac{b(z)}{a(z)} = \frac{1 + b_1 z + \cdots + b_q z^q}{1 - a_1 z - \cdots - a_p z^p},$$

and assume that numerator and denominator have no common roots. Let us firstly recall that $c(z)$ has the partial-fraction representation:

$$c(z) = G(z) + \sum_{j=1}^{s} G_j \left(\frac{1}{1 - \alpha_j z} \right),$$

where $1/\alpha_1,\ 1/\alpha_2,\ \ldots,\ 1/\alpha_s$ are the distinct roots of the equation

$$1 - a_1 z - \cdots - a_p z^p = 0, \tag{2.61}$$

G is a polynomial and

$$G_j \left(\frac{1}{1 - \alpha_j z} \right) = \frac{A_{j1}}{1 - \alpha_j z} + \cdots + \frac{A_{js_j}}{(1 - \alpha_j z)^{s_j}},$$

where s_j is the multiplicity of $1/\alpha_j$. For this standard algebraic result, see [5, pp. 84-86 and p. 110, Exercise 6], [1, pp. 30-32] and [2, pp. 258-264] for applications to integration of rational functions.

Secondly, computation of the coefficients of the Taylor expansion of

$$K(z) = (1 - \alpha z)^{-m},$$

m being a positive integer, gives

$$\begin{aligned} \mu_k &= \frac{(-\alpha)^k (-m)(-m-1)\cdots(-m-k+1)}{k!} \\ &= \frac{\alpha^k (k+m-1)\cdots(k+1)}{(m-1)!} \\ &= \tau_1 \alpha^k k^{m-1} + \tau_2 \alpha^k k^{m-2} + \cdots \tau_m \alpha^k . \end{aligned} \tag{2.62}$$

The function K is analytic in the region $|z| < |1/\alpha|$. Thus if $|\alpha| < 1$, by Proposition 2.15,

$$H(\mathbf{g}) = 1 + \mu_1 \mathbf{g} + \mu_2 \mathbf{g}^2 + \cdots ,$$

and

$$H(B) = 1 + \mu_1 B + \mu_2 B^2 + \cdots .$$

If $|\alpha| > 1$, then

$$\begin{aligned} K(\mathbf{g}) = (1 - \alpha \mathbf{g})^{-m} &= \frac{-\alpha^{-m} \mathbf{g}^{-m}}{(1 - \alpha^{-1} \mathbf{g}^{-1})^m} \\ &= -\alpha^{-m} \mathbf{g}^{-m} (1 + \mu_1 \mathbf{g}^{-1} + \mu_2 \mathbf{g}^{-2} + \cdots) \\ &= -\alpha^{-m} \mathbf{g}^{-m} - \alpha^{-m} \mu_1 \mathbf{g}^{-m-1} - \alpha^{-m} \mu_2 \mathbf{g}^{-m-2} - \cdots , \end{aligned}$$

so that

$$K(B) = -\alpha^{-m} F^m - \alpha^{-m} \mu_1 F^{m+1} - \alpha^{-m} \mu_2 F^{m+2} - \cdots$$

Lastly, $b(\mathbf{g})/a(\mathbf{g}) \in L^2([-\pi\ \pi])$ if and only if $|\alpha_j| \neq 1$ for $j = 1, \ldots, s$. If none of the zeros of (2.61) lies on the unit circle the function $b(\mathbf{g})/a(\mathbf{g})$ is continuous in $[-\pi\ \pi]$ and therefore belongs to $L^2([-\pi\ \pi])$. Now suppose, with no loss of generality, that $|\alpha_1| = 1$, so that $\alpha_1 = e^{i\theta_1}$. We have, expanding the cosine in Taylor series,

$$|1 - \alpha_1 e^{-i\theta}|^2 = 2[1 - \cos(\theta - \theta_1)] = 2(\theta - \theta_1)^2 \left[\frac{1}{2!} - \frac{1}{4!}(\theta - \theta_1)^2 + \cdots \right].$$

Since we have assumed that $a(z)$ and $b(z)$ have no roots in common, there exists a neighborhood W of θ_1 in $[-\pi\ \pi]$ such that for $\theta \in W$

$$\left| \frac{b(e^{-i\theta})}{a(e^{-i\theta})} \right|^2 \geq C \frac{1}{(\theta - \theta_1)^{2s_1}}$$

with $C > 0$. As the right-hand side has no finite integral, $|\alpha_1| = 1$ implies that $b(\mathbf{g})/a(\mathbf{g}) \neq L^2([-\pi\ \pi])$.

Summing up:

Proposition 2.16 Assume that the polynomials $a(z)$ and $b(z)$ have no roots in common. Then
(i) If $a(z)$ has unit-modulus roots, equation (2.52) has no solution.
(ii) If $a(z)$ has no unit-modulus roots then $a(B)$ is invertible and the solution of (2.52) is

$$y_t = H(B) u_t = a(B)^{-1} b(B) u_t .$$

Moreover, assuming that $|\alpha_j| < 1$ for $j = 1, \ldots, h$, and $|\alpha_j| > 1$ for $j = h+1, \ldots, s$, the function $H(B)$ can be obtained using the partial-fraction decomposition and expanding backward the fractions corresponding to $j \le h$, i.e. as series in B, and forward the series corresponding to $j > h$, so that $H(B) = H_1(B) + H_2(F)$, with

$$H_1(B) = 1 + \nu_1 B + \nu_2 B^2 + \cdots, \qquad H_2(B) = \nu_{-1} F + \nu_{-2} F^2 + \cdots,$$

and

$$y_t = \sum_{k=-\infty}^{\infty} \nu_k u_{t-k}.$$

(iii) If γ is real, positive and greater than the maximum of

$$|\alpha_1|, \ \ldots, \ |\alpha_h|, \ |\alpha_{h+1}^{-1}|, \ \ldots, \ |\alpha_s^{-1}|,$$

then

$$\lim_{n\to\infty} \frac{\nu_{|n|}}{\gamma^n} = 0.$$

(iv) If $|\alpha_j| < 1$ for all $j = 1, \ldots, s$, the solution is a linear combination of present and past values of u_t. Moreover, $\nu_0 = 1$:

$$y_t = u_t + \nu_1 u_{t-1} + \cdots.$$

PROOF. Statements (i) and (ii) have already been proved. For (iv), observe that

$$(\nu_0 + \nu_1 B + \cdots)(1 + b_1 B + \cdots + b_q B^q) = 1$$

and use Fact 3 to show that $\nu_0 = 1$. For (iii), observe that the coefficients ν_k are obtained as linear combinations of terms like $\alpha_j^k k^m$, for $j \le h$, and $\alpha_j^{-k} k^m$, for $j > h$.

Observation 2.18 Note that the roots of the polynomial in (2.61) are the reciprocals of those of the characteristic polynomial (2.24). The solutions of our ARMA are linear combinations of present and past values of u_t if all the roots of (2.24) are less than unity, or, equivalently, if all the roots of (2.61) are greater than unity. From now on we will invariably use equation (2.61) and therefore require 'big' roots to obtain an inverse of $a(B)$ in B.

Observation 2.19 The main results of Section 2.3—in particular, Proposition 2.7 and Exercise 2.6—have been independently re-obtained by means of the isomorphism $\Phi : H^u \to L^2([-\pi\ \pi])$, partial fractions, the Taylor expansion of $(1 - \alpha z)^{-m}$, and other fairly simple arguments regarding the functions of $L^2([-\pi\ \pi])$.

Observation 2.20 Assuming that $a(z)$ and $b(z)$ have no roots in common and using partial fractions it is easy to show that no root of the polynomial (2.61) can be absent from the solution of (2.52) in the ARMA case, which implies an alternative proof of Proposition 2.8.

Exercise 2.15 Apply partial fractions to find the solution of the equation in Example 2.6.

Observation 2.21 Factorize the polynomial $a(z)$ as

$$a(z) = (1 - \check{\alpha}_1 z)(1 - \check{\alpha}_2 z) \cdots (1 - \check{\alpha}_p z),$$

where the $\check{\alpha}$'s are not necessarily distinct, so that $\check{\alpha}_j = \alpha_1$ for $j = 1, \ldots, s_1$, etc. If none of the α's has unit modulus,

$$a(B)^{-1} = (1 - \check{\alpha}_1 B)^{-1}(1 - \check{\alpha}_2 B)^{-1} \cdots (1 - \check{\alpha}_p B)^{-1}.$$

Each term $(1 - \check{\alpha}_s B)^{-1}$ can then be expanded in B or F according to whether $\check{\alpha}_s$ is smaller or bigger than unity, while the product can be obtained using Fact 3. This method leads to the solution of the ARMA equation. However, the advantage of partial fractions is that they allow to determine the speed at which the coefficients of $a(B)^{-1}$ tend to zero.

Exercise 2.16 Apply the method of Observation 2.21 to solve the equation in Example 2.6.

Exercise 2.17 Apply the method of Observation 2.21 to show that if the coefficients of $a(B)$ are real then the expansion of $a(B)^{-1}$ has real coefficients.

Let us recall that standard textbook define ARMA processes as the solutions of equations $a(B)x_t = b(B)u_t$, where the polynomial $a(z)$ has no roots of modulus smaller or equal to 1, and the polynomial $b(z)$ has no roots of modulus smaller than 1. The following statement, whose proof is deferred to Section 2.6, reconciles our definition of an ARMA process with the standard one. Precisely, if y_t is the solution of $a(B)x_t = b(B)u_t$, then y_t is also the solution of an ARMA equation with the autoregressive and moving average roots fulfilling the standard conditions. This has the consequence that an ARMA process y_t does not identify a unique ARMA equation, so that the standard conditions on the roots act as identifying restrictions. This issue will be further discussed in Section 2.6.

Proposition 2.17 Assume that the polynomials $a(z)$ and $b(z)$ have no roots in common and that no root of $a(z)$ has unit modulus, so that the equation $a(B)x_t = b(B)u_t$ has the unique solution $y_t = [a(B)^{-1}]b(B)u_t$. Moreover, let

$$\begin{aligned} a(B) &= (1-\alpha_1 B)\cdots(1-\alpha_h)(1-\alpha_{h+1}B)\cdots(1-\alpha_p B) \\ b(B) &= (1-\beta_1 B)\cdots(1-\beta_s)(1-\beta_{s+1}B)\cdots(1-\beta_q B), \end{aligned}$$

with $\alpha_j < 1$ for $j \le h$, $\alpha_j > 1$ for $j > h$, $\beta_j \le 1$ for $j \le s$ and $\beta_j > 1$ for $j > s$ (note that we are not imposing that no root of $b(z)$ has unit modulus). There exists a white noise $w_t = c(B)u_t$ such that y_t is the unique solution of the ARMA equation

$$a'(B)x_t = b'(B)w_t,$$

where

$$\begin{aligned} a'(B) &= (1-\alpha_1 B)\cdots(1-\alpha_h)(1-\alpha_{h+1}^{-1}B)\cdots(1-\alpha_p^{-1} B) \\ b'(B) &= (1-\beta_1 B)\cdots(1-\beta_s)(1-\beta_{s+1}^{-1}B)\cdots(1-\beta_q^{-1} B). \end{aligned}$$

CASE 2. As we have seen, the equation $(1-B)x_t = u_t$ has no stationary solution. Heuristically, if we try to obtain $(1-B)^{-1}$ as the limit of $(1-\alpha B)^{-1}$, for $\alpha \to 1$, we see that all the coefficients of $(1-\alpha B)^{-1}$ tend to 1, so that in the limit we end up with a sequence that is not square summable (it is not even declining). Of course we do not find a stationary solution for $(1-B)^m x_t = u_t$, for m integer and greater than 1, whereas in the case $m = 0$ a solution trivially exists. This seems to be the end of the story as long as we confine ourselves to the space H^u and the operator B. However, when the equation $(1-B)^m x_t = u_t$ is transformed into $(1-\mathbf{g})^m f_t = \mathbf{g}^{-t}$, it is natural to consider non-integer values of m, values of m that lie between 0 and 1 in particular, and see whether a stationary solution exists for values of m that are positive and close to zero. As it turns out, a stationary solution exists for exponents that lie between 0 and $1/2$. Moreover, the solution is a moving average whose coefficients decline at a slower rate as compared to the moving averages resulting from ARMA equations.

As $0 < d < 1/2$, both functions $(1-\mathbf{g})^d$ and $(1-\mathbf{g})^{-d}$ belong to $L^2([-\pi\ \pi])$. For, the first is bounded while the squared modulus of the second tends to infinity as $\theta \to 0$ with the speed of θ^{-2d}. Application of Proposition 2.13 gives that the equation

$$(1-B)^d x_t = u_t \tag{2.63}$$

has the unique solution

$$y_t = (1-B)^{-d} u_t. \tag{2.64}$$

Note that the function $(1-\mathbf{g})^d$ is bounded but not bounded away from zero, so that the operator $(1-B)^d$ is not invertible. Thus although

$$(1-B)^d\left[(1-B)^{-d}u_t\right]=u_t,$$

the map $(1-B)^{-d}$ is defined only on $\check{H}^u$, i.e. it belongs to $\mathcal{A}(B)$ but not to $\mathcal{B}(B)$.

Now we want to prove that (2.63) is an infinite autoregressive equation, i.e. that $(1-B)^d$ has the one-sided representation

$$(1-B)^d = 1+\phi_1 B+\phi_2 B^2+\cdots$$

and that $(1-B)^{-d}$ has the one-sided representation

$$(1-B)^{-d} = 1+\psi_1 B+\psi_2 B^2+\cdots,$$

so that the solution y_t is a linear combination of present and past values of u_t. Moreover, we want to study the rate at which the coefficients ϕ_k and ψ_k tend to zero.

Let α be real, positive and less than unity and consider the complex function $K_\alpha(z)=(1-\alpha z)^d$. $K_\alpha(z)$ is analytic in the disk $|z|<\alpha^{-1}$ and has the Taylor expansion

$$K_\alpha(z)=1+\frac{-d}{1!}\alpha z+\cdots+\frac{(-d)(-d+1)\cdots(-d+k-1)}{k!}\alpha^k z^k+\cdots,$$

so that, by Proposition 2.15,

$$(1-\alpha\mathbf{g})^d = 1+\frac{-d}{1!}\alpha\mathbf{g}+\cdots+\frac{(-d)(-d+1)\cdots(-d+k-1)}{k!}\alpha^k\mathbf{g}^k+\cdots$$

is the Fourier expansion of $(1-\alpha\mathbf{g})^d$. Moreover, firstly, as $\alpha\to 1$, we have

$$|(1-\alpha e^{-i\theta})^d-(1-e^{-i\theta})^d|^2\to 0$$

for all $\theta\in[-\pi\ \pi]$. Secondly, using $|1-\alpha e^{-i\theta}|^2=1+\alpha^2-2\alpha\cos\theta$,

$$|(1-\alpha e^{-i\theta}|^2\le 4$$

for all $\theta\in[-\pi\ \pi]$ and $0<\alpha\le 1$, so that

$$\left|(1-\alpha e^{-i\theta})^d-(1-e^{-i\theta})^d\right|^2\le\left(|1-\alpha e^{-i\theta}|^d+|1-e^{-i\theta}|^d\right)^2\le 2^{2d+2}$$

for all $\theta \in [-\pi\ \pi]$ and $0 < \alpha \leq 1$. Thus the Lebesgue bounded convergence theorem can be applied and

$$\lim_{\alpha \to 1} \int_{-\pi}^{\pi} \left| (1 - \alpha e^{-i\theta})^d - (1 - e^{-i\theta})^d \right|^2 d\theta = 0$$

As a consequence, using Proposition 2.14,

$$(1 - \mathbf{g})^d = 1 + \frac{-d}{1!}\mathbf{g} + \cdots + \frac{(-d)(-d+1)\cdots(-d+k-1)}{k!}\mathbf{g}^k + \cdots$$

is the Fourier expansion of $(1 - \mathbf{g})^d$, that is,

$$(1 - B)^d = 1 + \frac{-d}{1!}B + \cdots + \frac{(-d)(-d+1)\cdots(-d+k-1)}{k!}B^k + \cdots .$$

Two easy modifications of the above reasoning lead to

$$(1 - B)^{-d} = 1 + \frac{d}{1!}B + \cdots + \frac{d(d+1)\cdots(d+k-1)}{k!}B^k + \cdots .$$

Everything goes as above, with $-d$ replacing d, except that, firstly, as $\alpha \to 1$,

$$(1 - \alpha z)^{-d} \to (1 - z)^{-d}$$

for $\theta \neq 0$, thus a.e. in $[-\pi\ \pi]$. Moreover,

$$|1 - \alpha e^{-i\theta}|^2 \geq (1 - \cos^2\theta)$$

for all $\theta \in [-\pi\ \pi]$ and $0 < \alpha \leq 1$ (just take the derivative of $1 + \alpha^2 - 2\alpha\cos\theta$ with respect to α). As a consequence

$$\begin{aligned} \left| (1 - \alpha e^{-i\theta})^{-d} - (1 - e^{-i\theta})^{-d} \right|^2 &\leq \left(|1 - \alpha e^{-i\theta}|^{-d} + |1 - e^{-i\theta}|^{-d} \right)^2 \\ &\leq \frac{4}{(1 - \cos\theta)^d (1 + \cos\theta)^d} \end{aligned}$$

for $0 < \alpha \leq 1$ and $\theta \neq 0$, where, as $d < 1/2$, the right hand side has finite integral. Summing up, the function $|(1 - \alpha e^{-i\theta})^{-d} - (1 - e^{-i\theta})^{-d}|^2$ tends to zero a.e. in $[-\pi\ \pi]$ and is bounded a.e. by an integrable function. This is sufficient to apply the Lebesgue bounded convergence theorem.

The asymptotic behavior of the coefficients of $(1 - B)^d$ and $(1 - B)^{-d}$ can be analysed by means of the Γ function and the Stirling's approximation formula. Let us recall that, for $x > 0$ (see e.g. [15], pp. 192-3),

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt.$$

The Γ function fulfills the property

$$\Gamma(x+1) = x\Gamma(x), \tag{2.65}$$

which can be used to extend Γ to non-positive real numbers, with the exception of 0 and all negative integers. An easy exercise using (2.65) shows that the coefficients ϕ_k and ψ_k can be written as

$$\phi_k = \frac{\Gamma(k-d)}{\Gamma(k+1)\Gamma(-d)}, \qquad \psi_k = \frac{\Gamma(k+d)}{\Gamma(k+1)\Gamma(d)}. \tag{2.66}$$

An approximation to $\Gamma(x)$, for x large, is provided by the Stirling's formula. Precisely (see [15], pp. 194-5):

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{x^x e^{-x}\sqrt{2\pi x}} = 1.$$

Another exercise shows that using the Stirling's approximation in (2.66) we obtain

$$\phi_k \sim \frac{k^{-d-1}}{\Gamma(-d)}, \qquad \psi_k \sim \frac{k^{d-1}}{\Gamma(d)},$$

the symbol '$\sim$' meaning that the ratio between left and right hand side tends to 1 as $k \to \infty$. Because $-d-1 < -1$ the sequence $\{\phi_k,\ k = 1, 2, \ldots\}$ is absolutely summable. By contrast, as $d - 1 > -1$, the sequence $\{\psi_k,\ k = 0, 1, \ldots\}$ is not absolutely summable, though, as $2(d-1) < -1$, it is square summable. Incidentally, note that $(1-B)^d$ can be defined for all $d > 0$ and belongs to $\mathcal{B}(B)$ (with absolute summability of the coefficients ϕ_k), but that $(1-B)^{-d}$ belongs to $\mathcal{A}(B)$ only if $d < 1/2$.

Quite obviously, we can start with $-1/2 < d < 0$ in equation (2.63), the solution being (2.64). As the case $d = 0$ is trivial, equation (2.63) has been solved for all d in the open interval $(-1/2\ \ 1/2)$.

In conclusion, for $-1/2 < d < 1/2$ the equation $(1-B)^d x_t = u_t$ has the solution $(1-B)^{-d} u_t$. Both $(1-B)^d$ and $(1-B)^{-d}$ are one-sided. Apart from $d = 0$, which is trivial, the coefficients of $(1-B)^{-d}$ tend to zero like k^{d-1}, and therefore slower than geometrically, slower, in particular, than the coefficients of ARMA processes.

In general, given the moving average

$$y_t = u_t + c_1 u_{t-1} + \cdots + c_s u_{t-s} + \cdots,$$

the absolute value $|c_s|$ can be taken as a measure of the memory that y_t has of the shock u_{t-s}, which has occurred s periods before. ARMA processes, for which

$|c_s|$ declines geometrically, are referred to as *short memory* processses, whereas processes for which $|c_s|$ declines with a hyperbolic law, like $(1 - B)^{-d} u_t$ with $0 < d < 1/2$, are referred to as *long memory* processes.

Summary. We have defined the operator $B : H^u \to H^u$. To every square-summable sequence there corresponds a map $b(B) : \check{H}^u \to H^u$. Maps $b(B)$ that are bounded can be extended to operators $b(B) : H^u \to H^u$. The autoregressive equation $a(B)x_t = u_t$ can be generalized to $a(B)x_t = b(B)u_t$, where $a(B)$ and $b(B)$ are both maps of $\check{H}^u$ into H^u. Then we have established the correspondence $\Phi : u_t \to \sqrt{\frac{\sigma_u}{2\pi}} e^{it(\cdot)}$ and shown that the linear extension of Φ to a map $\Phi : H^u \to L^2([-\pi\ \pi])$ is an isomorphism. Moreover, the map $b(B)$ corresponds, *via* Φ, to the multiplication map in $L^2([-\pi\ \pi])$, which maps the function f to $b(e^{-i(\cdot)})f$. Boundedness and invertibility of multiplication maps receive an easy and elegant characterization in $L^2([-\pi\ \pi])$, while the equation $a(B)x_t = b(B)u_t$ has a unique solution if and only if the function $[a(e^{-i(\cdot)})]^{-1} b(e^{-i(\cdot)})$ belongs to $L^2([-\pi\ \pi])$. The autoregressive equation, generalized to an ARMA equation, is solved using partial fractions and elementary Taylor expansions, thus obtaining with a different method the main results of Section 2.3. Lastly, the method is applied to the equation $(1 - B)^d x_t = u_t$, whose solution is shown to be a long memory process.