2.3 Autoregressive processes. Direct approach

We want to study the difference equation

$$x_t = a_1 x_{t-1} + \cdots + a_p x_{t-p} + u_t, \tag{2.8}$$

where u_t is a white noise process belonging to $L^2(\Omega, \mathcal{F}, P)$. We assume that

$$a_p \neq 0 \quad \text{and} \quad \sigma_u^2 \neq 0.$$

By a solution we mean a *process* $\{w_t,\ t \in \mathbb{Z}\}$, *belonging to* $L^2(\Omega, \mathcal{F}, P)$, such that (2.8) is fulfilled when x_t is replaced by w_t.

Analysis of equation (2.8), in which a stochastic variable is determined as a linear combination of its past values plus an uncorrelated "shock", hardly needs a motivation. Indeed, autoregressive processes naturally emerge in (a) micro- and macroeconomic analysis as a result of both models based on routine behaviors and models based on intertemporal optimization, and (b) modeling and forecasting of empirical processes.

Let us begin by stating a well known principle which is valid for all linear equations (the proof is elementary and is left to the reader).

Proposition 2.3 Let w_t be a solution to (2.8). Then all the solutions of (2.8) are obtained as $w_t + z_t$, where z_t is any process belonging to $L^2(\Omega, \mathcal{F}, P)$ which is a solution of the homogeneous equation

$$x_t = a_1 x_{t-1} + \cdots + a_p x_{t-p}. \tag{2.9}$$

Thus analysis of (2.8) breaks up into the search for a particular solution and the general solution of the homogeneous equation.

2.3.1 The case $p = 1$

Let us start with the first order equation and its companion homogeneous equation:

$$x_t = a x_{t-1} + u_t \tag{2.10}$$

$$x_t = a x_{t-1}. \tag{2.11}$$

As we will see, this problem contains in an elementary version all the notions that must be developed to deal with the general case.

Firstly observe that if z_t is a solution of (2.11) and $z_0 = A \in L^2(\Omega, \mathcal{F}, P)$, then $z_t = Aa^t$. Thus we only need a particular solution of (2.10). Let us further specify the problem as that of finding a weakly stationary solution. (Note that for

$|a| < 1$ a stationary solution has already been found, see the discussion of process (1.5) in Section 1.1.)

Assume that a weakly stationary solution of (2.10) exists and denote it by w_t. Consider the projection of w_t on the space H^u,

$$w_t = \text{Proj}(w_t|H^u) + R_t = \sum_{k=-\infty}^{\infty} b_{tk} u_{t-k} + R_t, \tag{2.12}$$

where $R_t \perp u_{t-k}$ for all $k \in \mathbb{Z}$, and $b_{tk} = \frac{w_t \cdot u_{t-k}}{\sigma_u^2}$. Since $u_{t-k} = w_{t-k} - aw_{t-k-1}$, so that $w_t \cdot u_{t-k} = w_t \cdot (w_{t-k} - aw_{t-k-1})$, and since w_t is stationary by assumption, $w_t \cdot u_{t-k} = E(w_t w_{t-k}) - aE(w_t w_{t-k-1})$ is independent of t, so that the last equation can be rewritten as

$$w_t = \sum_{k=-\infty}^{\infty} b_k u_{t-k} + R_t, \tag{2.13}$$

where b_k is square summable (by Proposition 1.4, $\sigma_u^2 \sum b_k^2$ is the squared norm of the projection of w_t on H^u). Substituting in (2.10),

$$\sum_{k=-\infty}^{\infty} (b_k - ab_{k-1}) u_{t-k} - u_t = -(R_t - aR_{t-1}).$$

Since $R_t \perp u_{t-k}$ for all k, this implies

$$\sum_{k=-\infty}^{\infty} (b_k - ab_{k-1}) u_{t-k} - u_t = 0 \tag{2.14}$$

$$R_t - aR_{t-1} = 0. \tag{2.15}$$

Since u_t is an orthogonal sequence (2.14) is equivalent to

$$b_k - ab_{k-1} = \begin{cases} 0 \text{ for } k \neq 0 \\ 1 \text{ for } k = 0. \end{cases} \tag{2.16}$$

This can be interpreted as a difference equation for sequences of real numbers. Note that (i) the left hand side is identical to the left hand side of (2.10), (ii) we are looking for solutions that are *square-summable* sequences. Forgetting for the moment square summability, the solutions of (2.16) are all obtained as one particular solution plus any solution of the homogeneous equation $b_k - ab_{k-1} = 0$. To determine the latter observe that if c is the value taken for $k = 0$, then ca^k is the value at any k. A particular solution of (2.16) is easy to find. Set

$$\check{b}_k = \begin{cases} 0 & \text{for } k < 0 \\ a^k & \text{for } k \geq 0, \end{cases}$$

so that the general solution is the sequence

$$b_k = \begin{cases} ca^k & \text{for } k < 0 \\ a^k + ca^k & \text{for } k \geq 0, \end{cases} \tag{2.17}$$

depending on the parameter c. Now we have to impose square summability. As the reader can easily check, (i) if $|a| < 1$, square summability is equivalent to $c = 0$; (ii) if $|a| > 1$, we must set $c = -1$; (iii) if $|a| = 1$ none of the solutions is square summable.

The above results are illustrated in Figure 2.3 for three real values of a. For $a = .8$ the solution is obtained by setting $c = 0$. For $a = 1.2$ the particular solution (starred) diverges and is therefore not square summable. Setting $c = -1$ the solution of the homogeneous equation corresponding to $k \geq 0$ (dotted) offsets the particular solution, so that the square-summable solution (circled) equals the solution of the homogeneous equation for $k < 0$, and equals zero for $k \geq 0$. Lastly, for $a = 1$ the particular solution (starred) is not square summable, but the only solution of the homogeneous equation offsetting the ones corresponding to $k \geq 0$ produces a non square-summable sequence for $k < 0$.

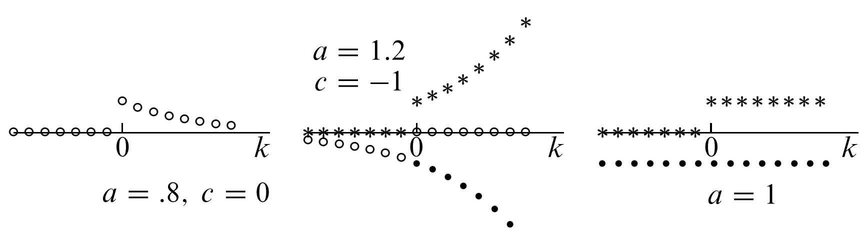

FIGURE 2.3

Now let us go back to (2.13). Observe firstly that stationarity of w_t and of the moving average at the right hand side, plus the condition $R_t \perp u_{t-k}$ for all k, imply that R_t must be stationary (orthogonality implies that the autocovariance function of w_t is the sum of the autocovariance functions of the two components at the right hand side). On the other hand, all the solutions of (2.15) take the form $R_t = Aa^t$, where $A \in L^2(\Omega, \mathcal{F}, P)$, which is not a stationary process if $|a| = 1$, unless $A = 0$. Thus if $|a| \neq 1$, the moving average at the left hand side of (2.13), where b_k is the unique square-summable solution of (2.16), is the only stationary solution of equation (2.10).

Proposition 2.4 Consider the difference equation (2.10). Then:
(i) If $|a| < 1$ the only stationary solution is

$$w_t = u_t + au_{t-1} + a^2 u_{t-2} + \cdots ,$$

and the general solution is $w_t + Aa^t$, where A is any stochastic variable belonging to $L^2(\Omega, \mathcal{F}, P)$.
(ii) If $|a| > 1$ the only stationary solution is

$$w_t = -a^{-1} u_{t+1} - a^{-2} u_{t+2} - \cdots ,$$

and the general solution is $w_t + Aa^t$, where A is any stochastic variable belonging to $L^2(\Omega, \mathcal{F}, P)$.
(iii) If $|a| = 1$, that is $a = e^{i\phi}$, there is no stationary solution. The process

$$w_t = \begin{cases} \sum_{s=0}^{t-1} a^s u_{t-s} \text{ if } t > 0 \\ 0 \text{ if } t = 0 \\ -\sum_{s=0}^{-t-1} a^{-s-1} u_{t-s+1} \text{ if } t < 0 \end{cases} \tag{2.18}$$

is a particular solution. The general solution is $w_t + Ae^{i\phi t}$, where A is any stochastic variable belonging to $L^2(\Omega, \mathcal{F}, P)$. Note that if $E(A) = 0$ the process $Ae^{i\phi t}$ is stationary (see Example 1.12).

PROOF. For (i) and (ii), we have already proved that w_t is the only stationary solution of (2.10). The general solution is obtained by adding a solution of equation (2.11). For (iii), the reader can easily check that the particular solution (2.18) fulfills equation (2.10). In the familiar case $a = 1$, the general solution is

$$w_t = \begin{cases} A + \sum_{s=0}^{t-1} u_{t-s} \text{ if } t > 0 \\ A \text{ if } t = 0 \\ A - \sum_{s=0}^{-t-1} u_{t-s+1} \text{ if } t < 0, \end{cases} \tag{2.19}$$

where A is any stochastic variable in $L^2(\Omega, \mathcal{F}, P)$. Note that $\text{var}(w_t) \to \infty$ for $|t| \to \infty$. The non-stationary process (2.19) is known as *random walk.*

Observation 2.4 The particular solution (2.18), with $w_0 = 0$ almost surely, has been chosen only for convenience. Obviously, considering for simplicity the case $a = 1$, the process

$$\begin{array}{rcl} & \vdots & \\ w_{t_0+2} & = & u_{t_0+1} + u_{t_0+2} \\ w_{t_0+1} & = & u_{t_0} \\ w_{t_0} & = & 0 \\ w_{t_0-1} & = & -u_{t_0} \\ w_{t_0-2} & = & -u_{t_0} - u_{t_0-1} \\ & \vdots & \end{array} \tag{2.20}$$

for any $t_0 \in \mathbb{Z}$, might have been selected. Note that this process can be obtained by suitably specifying A in (2.19).

Observation 2.5 The process (2.18) belongs to H^u but is not a moving average of u_t (it is not even stationary).

Observation 2.6 Note that the homogeneous equation (2.11) always has a trivial stationary solution, namely the process whose stochastic variables are zero with probability one. Note also that (2.11) has non trivial stationary solutions if and only if equation (2.10) has no stationary solutions, that is when $|a| = 1$.

Lastly, consider the equation

$$x_t = ax_{t-1} + c + u_t, \tag{2.21}$$

where $c \in \mathbb{R}$. If $|a| < 1$ the stationary solution is

$$z_t = \frac{c}{1-a} + u_t + au_{t-1} + \cdots ,$$

and the general solution is $z_t + Aa^t$. If $|a| > 1$ the stationary solution is

$$z_t = \frac{c}{1-a} - a^{-1}u_{t+1} - a^{-1}u_{t+1} - \cdots ,$$

and the general solution is $z_t + Aa^t$. If $|a| = 1$ a particular solution is $w_t + h_t$, where w_t is determined as in (2.18) and

$$h_t = \begin{cases} c\sum_{s=0}^{t-1} a^s \text{ if } t > 0 \\ 0 \text{ if } t = 0 \\ -c\sum_{s=0}^{-t-1} a^{-s-1} \text{ if } t < 0. \end{cases}$$

For $a = 1$ the general solution is

$$w_t = \begin{cases} A + ct + \sum_{s=0}^{t-1} u_{t-s} \text{ if } t > 0 \\ A \text{ if } t = 0 \\ A + ct - \sum_{s=0}^{-t-1} u_{t-s+1} \text{ if } t < 0, \end{cases}$$

which is known as *random walk with drift*.

2.3.2 The general autoregressive process

Let us begin with the homogeneous equation (2.9) and suppose that the solutions we are looking for are two-sided sequences of complex numbers $\{c_t,\ t \in \mathbb{Z}\}$. It is easily seen that such solutions form a vector subspace of the space of all sequences, i.e. that by summing two solutions or multiplying a solution by a complex number we obtain a solution; call V such a subspace. Moreover, given the values of a solution for $t = 0,\ 1,\ \dots,\ p-1$, the solution is determined for all t. To see this use equation (2.9) to determine the solution at $t = p$, then $t = p+1$, and so on. Then, since $a_p \neq 0$ by assumption, rewrite (2.9) as $x_{t-p} = a_p^{-1}(-x_t + a_1 x_{t-1} + \cdots + a_{p-1} x_{t-p+1})$, that is

$$x_t = a_p^{-1}(-x_{t+p} + a_1 x_{t+p-1} + \cdots + a_{p-1} x_{t+1}),$$

to determine the solution at $t = -1$, $t = -2$ and so on. Thus the dimension of V cannot be greater than p. On the other hand, the following are p independent solutions:

$$\begin{array}{ccccccccc} \cdots & t=-1 & t=0 & t=1 & \cdots & t=p-1 & t=p & \cdots \\ \cdots & a_{p-1}a_p^{-1} & 1 & 0 & \cdots & 0 & a_p & \cdots \\ \cdots & a_{p-2}a_p^{-1} & 0 & 1 & \cdots & 0 & a_{p-1} & \cdots \\ & \vdots & & & & & & \\ \cdots & -a_p^{-1} & 0 & 0 & \cdots & 1 & a_1 & \cdots \end{array} \qquad (2.22)$$

so that the dimension of V is exactly p.

Observation 2.7 Of course the "initial conditions" can be given for any sequence $t_0,\ t_0+1,\ \dots,\ t_0+p-1$ instead of $0,\ 1,\ \dots,\ p-1$, but not for any sequence $t_0,\ t_1,\ \dots,\ t_{p-1}$. For example, if equation (2.9) is $x_t - x_{t-2} = 0$, we cannot choose values for x_0 and x_2 as we like.

However, the p sequences (2.22) do not seem very useful if our problem is the behavior of a solution when $t \to +\infty$ or $t \to -\infty$. Much more promising sequences can be obtained by assuming, in analogy with the case $p = 1$, that α^t, with $\alpha \neq 0$, is a solution of (2.9), that is $\alpha^t - a_1\alpha^{t-1} - \cdots - a_p\alpha^{t-p} = 0$. This implies that

$$\alpha^{t-p}(\alpha^p - a_1\alpha^{p-1} - \cdots - a_p) = 0. \tag{2.23}$$

i.e. that α is a root of the polynomial

$$f(x) = x^p - a_1 x^{p-1} - \cdots - a_p, \tag{2.24}$$

which is known as the *characteristic polynomial* of the difference equation (2.9). As the converse is also obviously true, we conclude that α^t is a solution of (2.9) if and only if α is a root of its characteristic polynomial. If all the roots of $f(x)$ are of multiplicity one, i.e. if there are p roots α_j, with $\alpha_j \neq \alpha_k$ for $j \neq k$, then the p solutions α_j^t are independent. This is easily seen by putting on the columns of a $p \times p$ matrix the values taken by the solutions α_j^t at $t = 0,\ 1,\ \ldots,\ p-1$:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_p \\ \vdots & \vdots & & \vdots \\ \alpha_1^{p-1} & \alpha_2^{p-1} & \cdots & \alpha_p^{p-1} \end{pmatrix}.$$

This is well known as a Vandermonde matrix, whose determinant does not vanish if the α's are distinct (see, e.g., [12], p. 35). Thus all the solutions of equation (2.9) can be written as

$$c_1\alpha_1^t + c_2\alpha_2^t + \cdots + c_p\alpha_p^t,$$

and their asymptotic behavior can be easily studied.

In general however $f(x)$ has q *distinct* roots $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_q$, $q \leq p$, with multiplicities $r_1,\ r_2,\ \ldots,\ r_q$, such that $r_1 + r_2 + \cdots + r_q = p$, and

$$f(x) = (x - \alpha_1)^{r_1}(x - \alpha_2)^{r_2} \cdots (x - \alpha_q)^{r_q}. \tag{2.25}$$

Proposition 2.5 Assume that $f(x)$ admits factorization (2.25). Then
(i) The following sequences of complex numbers

$$t^{s_j}\alpha_j^t, \tag{2.26}$$

for $j = 1, \ldots, q$, and $s_j = 0, \ldots, r_j - 1$, are p solutions of equation (2.9).
(ii) The p solutions (2.26) are independent.
(iii) Let g_{ht}, $h = 1, \ldots, p$, be an ordering of the sequences (2.26). The sequences

$$c_1 g_{1t} + c_2 g_{2t} + \cdots + c_p g_{pt}, \tag{2.27}$$

where the c_h's are complex numbers, are solutions of (2.9); conversely, if z_t is a solution of (2.9), then z_t has the form (2.27) with the coefficients c_h uniquely determined by z_t.

We do not give a proof of this proposition. Note that (iii) is a consequence of (i) and (ii), given that the space V has dimension p. To provide an intuitive motivation for the emergence of solutions (2.26) when some of the roots are multiple, consider the case $p = 2$, that is

$$x_t = a_1 x_{t-1} + a_2 x_{t-2}, \tag{2.28}$$

and suppose that

$$x^2 - a_1 x - a_2 = (x - \alpha_1)(x - \alpha_2),$$

with $\alpha_1 \neq \alpha_2$. In this case the solutions (2.26) are α_1^t and α_2^t. Another couple of solutions is

$$\alpha_1^t, \quad \frac{\alpha_2^t - \alpha_1^t}{\alpha_2 - \alpha_1}.$$

The latter are independent as the matrix with their values at $t = 0, 1$ on the columns is

$$\begin{pmatrix} 1 & 0 \\ \alpha_1 & 1 \end{pmatrix}. \tag{2.29}$$

Now let $\alpha_2 \to \alpha_1$ (note that $a_1 = \alpha_1 + \alpha_2$ and $a_2 = -\alpha_1\alpha_2$, so that $\alpha_2 \to \alpha_1$ is obtained if $a_1 \to 2\alpha_1$ and $a_2 \to -\alpha_1^2$). We have:

$$\frac{\alpha_2^t - \alpha_1^t}{\alpha_2 - \alpha_1} = \frac{[\alpha_1 + (\alpha_2 - \alpha_1)]^t - \alpha_1^t}{\alpha_2 - \alpha_1} \to t\alpha_1^{t-1}. \tag{2.30}$$

Thus in the limit the sequences

$$\alpha_1^t, \quad t\alpha_1^t$$

emerge as candidate solutions (the second has been obtained by multiplying the limit in (2.30) by α_1). To see that $t\alpha_1^t$ is a solution of equation (2.28), for $\alpha_2 = \alpha_1$, remember that when the roots are coincident α_1 is a root of $f'(x)$ as well, that is

$$\begin{aligned} \alpha_1^2 - a_1\alpha_1 - a_2 &= 0 \\ 2\alpha_1 - a_1 &= 0. \end{aligned} \tag{2.31}$$

On the other hand, substituting $t\alpha_1^t$ into (2.28) gives

$$t\alpha_1^t - a_1(t-1)\alpha_1^{t-1} - a_2(t-2)\alpha_1^{t-2} = t\alpha_1^{t-2}(\alpha_1^2 - a_1\alpha_1 - a_2) + \alpha_1^{t-2}(a_1\alpha_1 + 2a_2).$$

Vanishing of the first term on the right hand side is ensured by the first equation on (2.31). For the second, note that $a_1\alpha_1 + 2a_2 = 0$ is obtained by summing the first line in (2.31), multiplied by -2, to the second multiplied by α_1.

Lastly, as the determinant (2.29) does not change as α_2 approaches α_1, the solutions α_1^t and $t\alpha_1^{t-1}$ are independent. Therefore α_1^t and $t\alpha_1^t$ are independent (but this can be seen directly of course).

Exercise 2.2 Consider the equation

$$x_t = 3ax_{t-1} - 3a^2x_{t-2} + a^3x_{t-3}.$$

The characteristic polynomial is $x^3 - 3ax^2 + 3a^2x - a^3 = (x-a)^3$, so that all the solutions are

$$c_1a^t + c_2ta^t + c_3t^2a^t.$$

Determine the solution x_t such that $x_0 = x_1 = x_2 = 1$.

For a proof of statements (i) and (ii) see [6], pp. 107-10. Another proof, requiring fairly hard but highly rewarding work, can be found in [11], Chapters 5 and 6 ([11] deals with differential equations but the analysis in Chapters 5 and 6 can be adapted with no relevant change to difference equations). The latter deserves a short informal description. To start with, define:

$$X_t = \begin{pmatrix} x_t \\ x_{t-1} \\ \vdots \\ x_{t-p+1} \end{pmatrix}, \quad A = \begin{pmatrix} a_1 & a_2 & \cdots & a_{p-1} & a_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \tag{2.32}$$

then transform (2.9) into the vector equation

$$X_t = AX_{t-1}. \tag{2.33}$$

Obviously the solution of (2.9) is nothing other than the first coordinate of the solution of (2.33). Therefore problem (2.9) becomes a particular case of the problem of solving a first-order linear system of difference equations. On the other hand, given X_0 (that is, given $x_0, x_{-1}, \ldots, x_{-p+1}$), the solution of 2.33 is $X_t = A^tX_0$, so that the behavior of all solutions of (2.33) can be analyzed by studying the powers of the matrix A. This can be done, as shown in [11], by using the Jordan form of A, which depends on the eigenvalues of A and their multiplicity (it is easily seen that the eigenvalues of A are the roots of the characteristic polynomial 2.24). To give an illustration let us assume that $p = 2$. If the roots are distinct the Jordan form is

$$A = B\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}B^{-1}, \tag{2.34}$$

so that

$$A^t = B\begin{pmatrix} \alpha_1^t & 0 \\ 0 & \alpha_2^t \end{pmatrix}B^{-1},$$

with the solutions of (2.9) emerging as linear combinations of α_1^t and α_2^t. On the other hand, if the roots are coincident the Jordan form is

$$A = B \begin{pmatrix} \alpha_1 & 0 \\ 1 & \alpha_1 \end{pmatrix} B^{-1} \tag{2.35}$$

(the matrix A cannot be diagonalized), so that

$$A^t = B \begin{pmatrix} \alpha_1^t & 0 \\ t\alpha_1^{t-1} & \alpha_1^t \end{pmatrix} B^{-1},$$

with the solutions of (2.9) emerging as linear combinations of α_1^t and $t\alpha_1^t$.

Exercise 2.3 In the two-dimensional case the matrix A defined in (2.32) is

$$\begin{pmatrix} a_1 & a_2 \\ 1 & 0 \end{pmatrix}.$$

Let α_1 and α_2 the eigenvalues of A, i.e. the roots of $x^2 - a_1 x - a_2$. If $v = (v_1\ v_2)'$ is an eigenvector of A corresponding to α_i, show that $v_1 = \alpha_i v_2$. As a consequence, if $\alpha_1 \neq \alpha_2$ there are two distinct eigenvectors of A, call them v and w, and

$$A \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} = \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix},$$

which is immediately transformed into (2.34). If $\alpha_1 = \alpha_2$ there is only one eigenvector, call it w. Letting v be any vector, independent of w, we have

$$Av = \beta v + \delta w, \qquad Aw = \alpha_1 w,$$

that is

$$A \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} = \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} \begin{pmatrix} \beta & 0 \\ \delta & \alpha_1 \end{pmatrix}.$$

Prove that $\beta = \alpha_1$. Moreover, show that you can choose a v, independent of w, in such a way that $\delta = 1$, thus obtaining (2.35), but not in such a way that $\delta = 0$.

The asymptotic behavior of the solutions (2.26) can be very easily described. Consider the sequence $t^m \alpha^t$, where m is a non-negative integer, and let γ be a positive real number. An elementary exercise shows that

$$\begin{aligned} \lim_{t \to +\infty} \frac{|t^m \alpha^t|}{\gamma^t} &= \begin{cases} 0 & \text{for } \gamma > |\alpha| \\ +\infty & \text{for } 0 < \gamma < |\alpha| \end{cases} \\ \lim_{t \to -\infty} \frac{|t^m \alpha^t|}{\gamma^t} &= \begin{cases} +\infty & \text{for } \gamma > |\alpha| \\ 0 & \text{for } 0 < \gamma < |\alpha| \end{cases} \end{aligned} \tag{2.36}$$

Thus, the sequences $t^m\alpha^t$ behave asymptotically almost in the same way as exponential sequences. For example, if $|\alpha| < 1$, the sequence $t^m\alpha^t$ tends to zero for $t \to +\infty$ faster that γ^t for any $\gamma > |\alpha|$, and slower than γ^t for any γ between 0 and $|\alpha|$.

Exercise 2.4 Use (2.36) to prove that the sequences

$$t^m\alpha^t, \quad t = 0,\ 1,\ 2,\ \ldots, \qquad t^m\beta^t, \quad t = 0,\ -1,\ -2,\ \ldots$$

are square summable if $|\alpha| < 1$ and $|\beta| > 1$ and are not square summable if $|\alpha| > 1$ and $|\beta| < 1$. Prove that if $|\gamma| = 1$ then neither of the sequences

$$t^m\gamma^t, \quad t = 0,\ 1,\ 2,\ \ldots, \qquad t^m\gamma^t, \quad t = 0,\ -1,\ -2,\ \ldots$$

is square summable.

Now let us go back to our original problem, that of finding the processes z_t belonging to $L^2(\Omega, \mathcal{F}, P)$ that fulfill equation (2.9). No difficulty arises to produce the stochastic-process version of the reasonings applied above to the sequences of real numbers solving (2.9). Firstly, a process solving (2.9) is completely determined by the values taken for $t = 0, \ldots, p-1$. Thus the vector space, call it $\mathcal{V}$, of all processes solving (2.9) has dimension p. Secondly, the sequences (2.26) can be interpreted as processes whose stochastic variables are constant with probability one. By Proposition 2.5, they are p linearly independent vectors of $\mathcal{V}$. The following statement is just as elementary as deriving (iii) from (ii) and (i) in Proposition 2.5.

Proposition 2.6 Given $A_k \in L^2(\Omega, \mathcal{F}, P)$, $k = 1, \ldots, p$, the stochastic process

$$A_1 g_{1t} + A_2 g_{2t} + \cdots + A_p g_{pt} \tag{2.37}$$

is a solution of (2.9). Conversely, if z_t is a solution of (2.9), then z_t has the form (2.37), with the stochastic variables A_k uniquely determined by z_t.

Having found all the solutions of equation (2.9), let us determine a particular solution of (2.8). Using the same argument of the previous section the reader will easily re-obtain equation (2.13). Substituting into (2.8),

$$\sum_{k=-\infty}^{\infty} (b_k - a_1 b_{k-1} - \cdots - a_p b_{k-p}) u_{t-k} - u_t = 0 \tag{2.38}$$

$$R_t - a_1 R_{t-1} - \cdots - a_p R_{t-p} = 0, \tag{2.39}$$

and therefore

$$b_k - a_1 b_{k-1} - \cdots - a_p b_{k-p} = \begin{cases} 0 \text{ for } k \neq 0 \\ 1 \text{ for } k = 0. \end{cases} \tag{2.40}$$

We look for square summable solutions of (2.40). If $\check{b}_k$ is a particular solution, the general solution of (2.40) is

$$b_k = \check{b}_k + [c_1 g_{1k} + c_2 g_{2k} + \cdots + c_p g_{pk}].$$

A particular solution $\check{b}_t$ is determined as follows. Let

$$B_k = d_1 g_{1k} + d_2 g_{2k} + \cdots + d_p g_{pk} \tag{2.41}$$

be the solution of the homogeneous equation (2.9) such that

$$B_0 = 1,\ B_{-1} = 0,\ \ldots,\ B_{-p+1} = 0$$

(obviously taking $0,\ -1,\ \ldots,\ -p+1$ instead of $0,\ 1,\ \ldots,\ p-1$ is immaterial, see Observation 2.7). Then

$$\check{b}_k = \begin{cases} 0 & \text{if } k < 0 \\ B_k & \text{if } k \geq 0, \end{cases}$$

so that the general solution is

$$b_k = \begin{cases} c_1 g_{1k} + \cdots + c_p g_{pk} \text{ if } k < 0 \\ (d_1 + c_1) g_{1k} + \cdots + (d_p + c_p) g_{pk} \text{ if } k \geq 0. \end{cases} \tag{2.42}$$

Now assume that none of roots of the characteristic polynomial has unit modulus. To determine c_h, remember that the solution g_{hk} equals $k^{s_j} \alpha_j^k$ for some α_j and $s_j < r_j$. Using Exercise 2.4, it easy to show that the only square-summable solution is obtained by setting

$$c_h = \begin{cases} 0 & \text{if } |\alpha_j| < 1 \\ -d_h & \text{if } |\alpha_j| > 1. \end{cases} \tag{2.43}$$

To see this remember that the sum of square summable sequences is square summable (see Section 1.3.8), see also Exercise 2.6 below). Another easy consequence of Exercise 2.4 is that equation (2.39) has no non-trivial stationary solution. Thus equation (2.8) has exactly one stationary solution. In conclusion

Proposition 2.7 Assume that none of the roots of the characteristic equation has unit modulus. Then equation (2.8) has exactly one stationary solution

$$w_t = \sum_{k=-\infty}^{\infty} b_k u_{t-k},$$

where the coefficients b_k are determined by (2.42), with d_k determined in (2.41) and c_k in (2.43). The general solution of (2.8) is

$$w_t + A_1 g_{1t} + A_2 g_{2t} + \cdots + A_p g_{pt},$$

where the A_h's are any stochastic variables belonging to $L^2(\Omega, \mathcal{F}, P)$.

Example 2.6 Consider the equation

$$x_t = 2.5x_{t-1} - x_{t-2} + u_t,$$

which has characteristic equation $x^2 - 2.5x + 1$. The roots are 2 and 0.5. Setting $B_k = 0.5^k d_1 + 2^k d_2$, we find $d_1 = -1/3$ and $d_2 = 4/3$. Thus

$$b_k = \begin{cases} -(4/3)2^k & \text{if } k < 0 \\ -(1/3)0.5^k & \text{if } k \geq 0, \end{cases}$$

and the stationary solution is

$$\begin{aligned} w_t &= -(1/3)\left(u_t + 0.5u_{t-1} + 0.5^2 u_{t-2} + \cdots\right) \\ &\quad -(4/3)\left(0.5u_{t+1} + 0.5^2 u_{t+2} + \cdots\right). \end{aligned}$$

Exercise 2.5 (Real Coefficients) Show that if the coefficients of (2.8) are real then the moving average w_t has real coefficients. In particular, determine the coefficients of w_t when $p = 2$ and the roots of the characteristic equation are complex (and conjugate), both when their modulus is greater and when it is smaller than unity. Hint: If $p = 2$ and the roots are complex conjugate then

$$g_{1t} = \rho^t e^{i\phi t} = \rho^t(\cos\phi t + i\sin\phi t), \qquad g_{2t} = \rho^t e^{-i\phi t} = \rho^t(\cos\phi t - i\sin\phi t).$$

Another set of independent solutions, that are *real*, is

$$\frac{g_{1t} + g_{2t}}{2} = \rho^t \cos\phi t, \qquad \frac{g_{1t} - g_{2t}}{2i} = \rho^t \sin\phi t.$$

Exercise 2.6 Assume, with no loss of generality, that the moduli of $\alpha_1, \ldots \alpha_m$ are smaller than 1, while the moduli of $\alpha_{m+1}, \ldots, \alpha_q$ are greater than 1. Let τ be the maximum among the numbers

$$|\alpha_1|, \ldots, |\alpha_m|, |\alpha_{m+1}|^{-1}, \ldots, |\alpha_q|^{-1}.$$

Prove that the coefficients b_k decline faster than $\gamma^{|k|}$, that is

$$\lim_{k\to+\infty} \frac{b_k}{\gamma^k} = 0, \quad \lim_{k\to-\infty} \frac{b_k}{\gamma^{|k|}} = 0,$$

for any γ such that $\tau < \gamma < 1$. Equivalently, prove that there exists $C_\gamma > 0$ such that $|b_k| \leq C_\gamma \gamma^{|k|}$, for any γ such that $\tau < \gamma < 1$.

Exercise 2.7 Prove that if all the roots of the characteristic equation are smaller than 1 in modulus, then the solution is one sided in the present and past of u_t:

$$x_t = u_t + b_1 u_{t-1} + \cdots$$

(note that $b_0 = 1$); if all the roots are greater than 1 in modulus, then the solution is one sided in the future of u_t:

$$x_t = b_{-1} u_{t+1} + b_{-2} u_{t+2} + \cdots .$$

In all other cases the solution is two sided.

Proposition 2.8 If at least one of the roots of the characteristic equation has unit modulus then equation (2.8) has no stationary solutions.

PROOF. Let us go back to the sequence B_k, i.e. to the solution of equation (2.9) with the condition

$$B_0 = 1, \ B_{-1} = 0, \ \ldots, \ B_{-p+1} = 0.$$

The coefficients d_h are determined by

$$\begin{pmatrix} g_{10} & \cdots & g_{p0} \\ g_{1,-1} & \cdots & g_{p,-1} \\ \vdots & & \vdots \\ g_{1,-p+1} & \cdots & g_{p,-p+1} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_p \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

that is

$$\begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_p \end{pmatrix} = \begin{pmatrix} g_{10} & \cdots & g_{p0} \\ g_{1,-1} & \cdots & g_{p,-1} \\ \vdots & & \vdots \\ g_{1,-p+1} & \cdots & g_{p,-p+1} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} D_{11} \\ D_{21} \\ \vdots \\ D_{p1} \end{pmatrix},$$

where the D_{ij}'s are the entries of the inverse matrix appearing above. Now, assume that $g_{1k} = t^{r_1-1}\alpha_1^t$ (so that if α_1 is simple $g_{1k} = \alpha_1^t$). The entry D_{11} is a ratio whose numerator is the determinant of

$$M = \begin{pmatrix} g_{2,-1} & \cdots & g_{p,-1} \\ \vdots & & \vdots \\ g_{2,-p+1} & \cdots & g_{p,-p+1} \end{pmatrix}.$$

Considering the difference equation whose characteristic polynomial is

$$\frac{f(x)}{x-\alpha_1} = (x-\alpha_1)^{r_1-1}(x-\alpha_2)^{r_2}\cdots(x-\alpha_q)^{r_q},$$

its solutions (2.26), for $k = -1, -2, \ldots, -p+1$, are the columns of M. Thus $\det(M) \neq 0$ and $D_{11} \neq 0$. The above reasoning shows that no root of the characteristic polynomial can be absent from solution (2.41), and therefore from $\check{b}_k$. But if a root has unit modulus it is impossible to get rid of it by appropriately choosing c_h in (2.43) (see also the case $a = 1$ in Figure 2.3). The proof is complete.

Exercise 2.8 Generalize Observation 2.6: Equation (2.9) has non-trivial stationary solutions if and only if equation (2.8) has no stationary solutions, that is, if and only if the characteristic equation has some unit roots. In that case the stationary solutions of (2.9) have the form 1.35:

$$A_1 e^{i\phi_1 t} + A_2 e^{i\phi_2 t} + \cdots + A_n e^{i\phi_n t},$$

where the numbers $e^{i\phi_j}$ are the unit modulus roots of the characteristic equation (see Exercise 1.26), and the stochastic variables A_j are zero-mean and mutually orthogonal.

Exercise 2.9 Consider the equation

$$x_t = 2x_{t-1} - x_{t-2}.$$

Determine its general solution. Observe that the equation can be rewritten as

$$(x_t - x_{t-1}) = (x_{t-1} - x_{t-2}) + u_t,$$

so that you can use (2.19) to determine $x_t - x_{t-1}$.

2.3.3 Autoregressive equations with initial or "final" conditions

Up until now we have studied solutions of equation (2.8) for $t \in \mathbb{Z}$ and established the conditions under which a stationary solution exists. Let us now again consider the autoregressive equation

$$x_t = a_1 x_{t-1} + \cdots a_p x_{t-p} + u_t, \tag{2.44}$$

where $\{u_t,\ t \in \mathbb{Z}\}$ is a white noise belonging to $L^2(\Omega, \mathcal{F}, P)$, but limit the validity of (2.44) to $t \geq t_0 + p$, so that a solution is a sequence

$$y_{t_0},\ y_{t_0+1},\ \ldots,\ y_{t_0+p},\ y_{t_0+p+1},\ \ldots$$

Propositions 2.3 and 2.6 apply with no modifications, so that all the solutions of (2.44) are

$$w_t + A_1 g_{1t} + A_2 g_{2t} + \cdots + A_p g_{pt},$$

where w_t is a particular solution. For the latter set, for $t \geq t_0$,

$$w_t = \sum_{k=-\infty}^{t-t_0} b_k u_{t-k}, \tag{2.45}$$

where the coefficients b_k are determined as in (2.42), with the coefficients c_h such that $\lim_{k\to-\infty} b_k = 0$. Note that here the coefficients b_k are not necessary unique (they are unique in the previous section under the assumption that no roots has unit modulus and that the coefficients c_h are determined by (2.43)). For example, if

$$x_t = 2x_{t-1} + u_t,$$

then, for $t \geq t_0$,

$$w_{1t} = u_t + 2u_{t-1} + \cdots + 2^{t-t_0} u_{t_0} \tag{2.46}$$

is a particular solution as well as

$$w_{2t} = -\frac{1}{2}u_{t+1} - \frac{1}{4}u_{t+2} + \cdots .$$

To obtain a unique solution the problem must be further specified. Here are two important cases.

CASE 1. Backward solution with given initial conditions.
The stochastic variables

$$x_{t_0},\ x_{t_0+1},\ \cdots,\ x_{t_0+p-1}$$

are *given* (in particular they may be constant with probability one) and *the* solution results by iterating (2.44):

$$\begin{aligned} x_{t_0+p} &= u_{t_0+p} + [a_1 x_{t_0+p-1} + \cdots + a_p x_{t_0}] \\ x_{t_0+p+1} &= u_{t_0+p+1} + [a_1 x_{t_0+p} + \cdots + a_p x_{t_0}] \\ &= u_{t_0+p+1} + a_1 u_{t_0+p} + [(a_1^2 + a_2) x_{t_0+p-1} + \cdots] \\ &\vdots \end{aligned}$$

This solution can also be obtained by choosing $c_h = 0$ for all $h = 1, 2, \ldots, p$, so that the coefficients b_k are zero for $k < 0$ (solution (2.46) in the example above). Once w_t is determined in this way, the stochastic variables A_h must be chosen such that

$$x_{t_0+s} - w_{t_0+s} = A_1 g_{1,t_0+s} + \cdots + A_p g_{p,t_0+s},$$

for $s = 0, 1, \ldots, p-1$ (this system has a unique solution by Proposition 2.5, (ii)). If all the roots of the characteristic equation are smaller than unity in modulus, the solution just obtained is “asymptotically stationary”, this meaning that the term $A_1 g_{1t} + \cdots + A_p g_{pt}$, which is called the *transient* in this case, vanishes as $t \to \infty$, while the difference between

$$w_t = \sum_{k=0}^{t-t_0} b_k u_{t-k}$$

and the stationary $\sum_{k=0}^{\infty} b_t u_{t-k}$ converges to zero. In the simple example

$$x_t = \alpha x_{t-1} + u_t,$$

the backward solution with initial condition x_{t_0} is

$$x_t = u_t + \alpha u_{t-1} + \cdots + \alpha^{t-t_0-1} u_{t_0+1} + x_{t_0} \alpha^{t-t_0}.$$

For $\alpha = 1$ we find the standard random walk with initial condition

$$x_t = u_t + \cdots + u_{t_0+1} + x_{t_0},$$

in which the u’s contribute to the value of x_t with the same weight, irrespective of whether they have occurred in the recent or in the remote past. For $|\alpha| < 1$ the term $x_{t_0} \alpha^{t-t_0}$ vanishes asymptotically, and the solution tends to the stationary $\sum_{k=0}^{\infty} \alpha^k u_{t-k}$, in which the weight of remote u’s dwindles as the powers of α. For $|\alpha| > 1$, as soon as x_t becomes big in absolute value as compared to the standard deviation of u_t the variation $x_{t+1} - x_t = (\alpha - 1) x_t + u_{t+1}$ is dominated by $(\alpha - 1) x_t$ and the pattern of x can no longer be distinguished from that of $B\alpha^t$.

CASE 2. Final conditions: Asymptotically stationary solutions.
Assume that no root of the characteristic equation has unit modulus. If the coefficients c_h are determined by (2.43), the particular solution w_t is asymptotically stationary. Moreover, set $A_h = 0$ if the root in g_{ht} is greater than 1 in modulus. With no loss of generality suppose that the roots in g_{ht} are smaller than 1 in modulus for $h \geq m$ and greater than 1 in modulus for $h > m$. Then the solutions

$$A_1 g_{1t} + \cdots + A_m g_{mt} + w_t$$

are asymptotically stationary. If all the roots of the characteristic equation are smaller than 1 in modulus we are back in Case 1 and p initial conditions are needed to determine the solution. Otherwise we need m initial conditions. A useful exercise is the analysis of the autoregressive equation $x_t = a_1 x_{t-1} + a_2 x_{t-2} + u_t$ when one of the roots has modulus smaller that 1 and the other greater.

Summary. The difference equation $x_t = a_1 x_{t-1} + \cdots + x_{t-p} + u_t$, where u_t is a white noise process, has a stationary solution if and only if no root of the characteristic polynomial $x^p - a_1 x_{p-1} - \cdots - a_p$ has unit modulus. Equivalently, no stationary solution exists if and only if at least one of the roots has unit modulus. In the no unit-root case the stationary solution is unique and is a moving average of u_t. The coefficients of the moving average decline exponentially faster than $\gamma^{|k|}$, where γ is any positive number smaller than 1 and greater than (a) the modulus of all the roots smaller than 1 in modulus and (b) the modulus of all the reciprocals of the roots greater than 1 in modulus. The moving average is one sided and backward (forward) if and only if all the roots are smaller than 1 (greater than 1) in modulus.