2.2 Moving Averages of a White Noise

Given the white noise $\{u_t,\ t \in \mathbb{Z}\}$, with $u_t \in L^2(\Omega, \mathcal{F}, P)$, let

$$H^u = \overline{\text{sp}}(u_t,\ t \in \mathbb{Z}).$$

By definition a white noise process has zero mean. Thus the condition $\gamma_k = \text{cov}(u_t, u_{t-k}) = u_t \cdot u_{t-k} - E(u_t)^2 = 0$, for $k \neq 0$, is identical to $u_t \perp u_{t-k} = 0$, for $k \neq 0$, so that the sequence u_t/σ_u is an orthonormal sequence in H^u. By Proposition 1.4, if $x \in H^u$,

$$x = \sum_{k=-\infty}^{\infty} a_{xk} u_{-k}, \quad a_{xk} = \frac{x \cdot u_{-k}}{\sigma_u^2}, \tag{2.4}$$

where $\sum_{k=-\infty}^{\infty} |a_{xk}|^2 < \infty$ (the use of "$a_{xk}u_{-k}$", instead of "$a_{xk}u_k$" is only a convenience, leading to the usual expression for moving averages; see below). Conversely, given the square-summable sequence a_k, the series $\sum_{k=-\infty}^{\infty} a_k u_{-k}$ converges in H^u. Calling y its limit we have $a_k = \frac{y \cdot u_{-k}}{\sigma_u}$. The function $\Psi : H^u \to \ell^2(\infty, \infty)$ defined as

$$\Psi(x) = \{\sigma_u a_{xk},\ k \in \mathbb{Z}\}, \tag{2.5}$$

is an isomorphism of H^u onto $l^2(-\infty, \infty)$ (see Proposition 1.5).

Definition 2.2 Given a square-summable sequence $\{a_k,\ k \in \mathbb{Z}\}$, the process

$$x_t = \sum_{k=-\infty}^{\infty} a_k u_{t-k},\ \ t \in \mathbb{Z}. \tag{2.6}$$

is called a *moving average of u_t*.

As in (1.3) the coefficients are independent of t. Here however the moving average is in general infinite. By (1.21), the autocovariance function of x_t is easily seen to be

$$\gamma_s = \sigma_u^2 \sum_{k=-\infty}^{\infty} a_k \overline{a_{k-s}}.$$

Exercise 2.1 If the coefficients a_s are real the autocovariances γ_k are real irrespective of whether the white noise u_t is real or not. The converse is false. Consider the process

$$x_t = u_t + \left(\overline{\alpha} - \frac{1}{\alpha}\right)(u_{t-1} + \overline{\alpha} u_{t-2} + \overline{\alpha}^2 u_{t-3} + \cdots),$$

where α is any complex number such that $|\alpha| < 1$. Prove that x_t is white noise, so that (1) its autocovariance function is real even when the coefficients of the moving average are not real, (2) a non trivial moving average of a white noise can be a white noise.

We will need the following definition.

Definition 2.3 The processes w_t and z_t, on $(\Omega, \mathcal{F}, P)$, are *costationary* if both are weakly stationary and $w_t \cdot z_{t-s} = E(w_t \overline{z_{t-s}})$ does not depend on t. When w_t and z_t are costationary we set $\gamma_s^{wz} = w_t \cdot z_{t-s}$. (Show that $E(z_t \overline{w_{t-s}}) = \overline{\gamma_{-s}^{wz}}$, and is therefore independent of t, so that we can set $\gamma_s^{zw} = z_t \cdot w_{t-s}$.)

Given the moving average $y_t = \sum_{k=-\infty}^{\infty} b_k u_{t-k}$, the processes x_t, as defined in (2.6), and y_t are costationary. For, applying (1.21), it is easily seen that

$$\gamma_s^{xy} = \sigma_u^2 \sum_{k=-\infty}^{\infty} a_k \overline{b_{k-s}}.$$

Conversely, suppose that $\{z_t,\ t \in \mathbb{Z}\}$ belongs to H^u. Applying (2.4), but omitting z in the coefficients for simplicity,

$$z_t = \sum_{k=-\infty}^{\infty} a_{tk} u_{-k}. \tag{2.7}$$

Now assume that z_t and u_t are costationary. This implies that

$$E(z_t \overline{u_{t-s}}) = a_{t,s-t}$$

does not depend on t. Defining $b_s = a_{t,s-t}$, so that $a_{tk} = b_{k+t}$, (2.7) can be rewritten as

$$z_t = \sum_{k=-\infty}^{\infty} a_{tk} u_{-k} = \sum_{k=-\infty}^{\infty} b_{k+t} u_{-k} = \sum_{h=-\infty}^{\infty} b_h u_{t-h}.$$

In conclusion:

Proposition 2.1 Moving averages of u_t, as defined in (2.6), belong to H^u and are costationary with each other, and with u_t in particular. Conversely, if $\{z_t,\ t \in \mathbb{Z}\}$ belongs to H^u and is costationary with u_t, then z_t is a moving average of u_t.

Example 2.5 Consider the process $w_t = u_{2t}$. Obviously w_t is a white noise blonging to H^u. However, $u_0 \cdot w_0 = \sigma_u^2$ whereas $u_t \cdot w_t = u_t \cdot u_{2t} = 0$ for $t \neq 0$. Thus w_t and u_t are not costationary, so that w_t cannot be represented as a moving average of u_t. Other examples are $y_t = u_{-t}$, and $z_t = u_0 e^{i\phi t}$.

Note that given $x \in H^u$ there exists a moving average of u_t whose range (see p. 3) contains x. To see this consider again (2.4) and define $z_t = \sum_{k=-\infty}^{\infty} a_{xk} u_{t-k}$, so that $x = z_0$. Obviously z_t is not the only moving average containing x; taking $w_t = z_{t-1}$ we would have $y = w_1$. However, it is the only moving average such that x corresponds to the integer 0 (check this). Thus H^u is *covered* by the ranges of $\{y_t,\ t \in \mathbb{Z}\}$, y_t being any moving average of u_t. As an exercise, the reader may prove that if y_t and z_t are moving averages of u_t then either the range of $\{y_t,\ t \in \mathbb{Z}\}$ and the range of $\{z_t,\ t \in \mathbb{Z}\}$ are equal, or their intersection is empty, and that the first alternative holds if and only if $z_t = y_{t-s}$ for an integer s.

Figure 2.1 provides a stylized picture of the spaces and subsets we are dealing with. The space H^u, which is mapped onto $l^2(-\infty, \infty)$ by the isomorphism Ψ, is partitioned into subsets, each constituting the range of a moving average of u_t. In the figure the range of u_t itself and that of z_t are represented and denoted by $[u_t]$ and $[z_t]$ respectively. Different ranges do not intersect and the union of all ranges coincides with H^u. Let us underline again that many different processes produce the same range, like z_t and $w_t = z_{t-1}$.

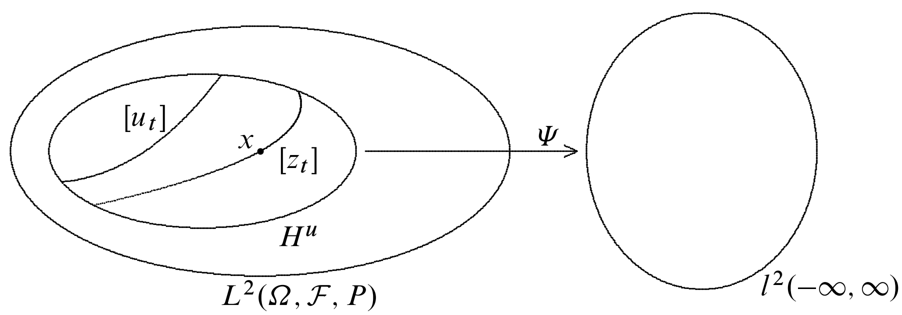

FIGURE 2.1

A very important statement regarding processes that are moving averages of a white noise is the following.

Proposition 2.2 Let u_t be a white-noise process, $x_t = \sum_{k=-\infty}^{\infty} a_k u_{t-k}$, and let γ_k be the autocovariance function of x_t. Then:
(i) $\lim_{s\to\infty} |\gamma_s| = 0$.
(ii) $|\gamma_s| < |\gamma_0| = \sigma_x^2$.

PROOF. Without loss of generality assume that $s \geq 0$. Let $a = \{a_k,\ k \in \mathbb{Z}\}$ and put $a_{(h)} = \{a_{k-h},\ k \in \mathbb{Z}\}$. Using a notation similar to that introduced in Section 1.3.8, if $b \in l^2(-\infty, \infty)$, let

$$b_k^{\{h\}} = \begin{cases} b_k & \text{if } k \leq h \\ 0 & \text{if } k > h. \end{cases}$$

Note that $b^{\{h\}} \cdot (c - c^{\{h\}}) = 0$, for all b and c, and that $b_{(n)}^{\{h\}} = b^{\{h-n\}}$. Letting m be the smallest integer greater or equal to $s/2$, we have, using the Cauchy-Schwartz inequality and the observations above,

$$\begin{aligned} a \cdot a_{(s)} &= [a^{\{m\}} + (a - a^{\{m\}})] \cdot [a_{(s)}^{\{m\}} + (a_{(s)} - a_{(s)}^{\{m\}})] \\ &= a^{\{m\}} \cdot a_{(s)}^{\{m\}} + (a - a^{\{m\}}) \cdot (a_{(s)} - a_{(s)}^{\{m\}}) \\ &= a^{\{m\}} \cdot a^{\{m-s\}} + (a - a^{\{m\}}) \cdot (a_{(s)} - a_{(s)}^{\{m\}}) \\ &\leq \|a\| \left(\|a^{\{m-s\}}\| + \|a - a^{\{m\}})\| \right). \end{aligned}$$

Since a is square summable, given $\epsilon > 0$, we can choose s_ϵ such that if $s > s_\epsilon$, the tails $a^{\{m-s\}}$ and $a - a^{\{m\}}$ are smaller than $\frac{\epsilon}{2\|a\|}$ in modulus. Statement (i) follows from the observation that $\gamma_s = \sigma_u^2 (a \cdot a_{(s)})$.

Regarding (ii), suppose that $|\gamma_h / \gamma_0| = 1$ for an integer $h > 0$. Then it easily seen that $x_{t-h} = \frac{\gamma_h}{\gamma_0} x_t$, and therefore

$$x_{t-sh} = \frac{\gamma_h^s}{\gamma_0^s} x_t = \frac{\gamma_{sh}}{\gamma_0} x_t,$$

so that $|\gamma_{sh}| = |\gamma_0|$, which contradicts statement (i).

Very often a formal proof is much more complicated than the underlying idea. For example, statement (i) in Proposition 2.2 can be completely grasped with the aid of Figure 2.2, the dotted line representing a, the starred line $a_{(13)}$, so that $m = 7$. As one immediately realizes, when s is large enough, the "belly" of a corresponds to the left tail of $a_{(s)}$, while the belly of $a_{(s)}$ corresponds to the right tail of a. The larger s the smaller the norm of the tails.

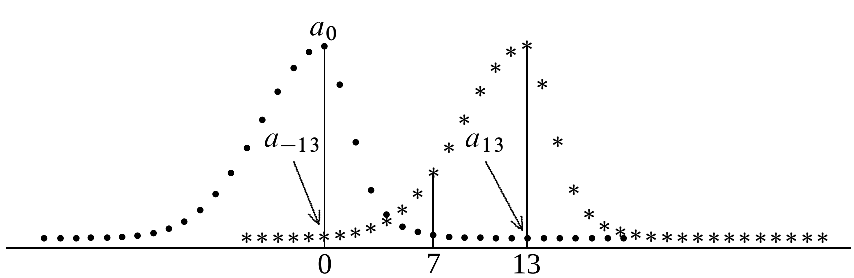

FIGURE 2.2

As we have seen, there exist weakly stationary processes belonging to H^u that are not moving averages of u_t. Some of them however are moving averages of other white noises (as a matter of fact, some of the processes in Example 2.5 are white noise). In general, when a weakly stationary process z_t is not defined as the moving average of a white noise, we may ask the question whether there exists a process $w_t = \sum_{k=-\infty}^{\infty} a_k v_{t-k}$, with v_t white noise on some probability space, such that the autocorrelation functions of z_t and of w_t coincide. Proposition 2.2 provides a necessary condition for a positive answer. Thus, for example, the answer is negative for the process $z_t = Ae^{i\phi t}$ (see Example 1.12), whose autocovariance function does not decay for $s \to \infty$.

Observation 2.2 Dropping the condition $E(u_t) = 0$ from the definition of white noise has the consequence that the infinite moving average with coefficients a_k makes sense only if $\sum_{k=-m}^{m} a_k$ converges (as we have seen in Observation 1.7, the mean of $\sum_{k=-m}^{m} a_k u_{t-k}$ should converge to the mean of the limit). This is a stricter condition as compared to square summability; for example, the sequence

$$a_k = \begin{cases} \dfrac{1}{k+1} \text{ for } k \geq 0 \\ 0 \text{ for } k < 0 \end{cases}$$

is square summable but not summable, see Observation 1.4.

Observation 2.3 A consequence of $E(u_t) = 0$ is that all the variables in H^u have zero mean. As on p. 43, denote by **1** the function of $L^2(\Omega, \mathcal{F}, P)$ which associates the real number 1 with every $\omega \in \Omega$ (the unit constant). Then define $\check{H}^u = \overline{\text{sp}}(\mathbf{1}, u_t,\ t \in \mathbb{Z})$. Note that $E(u_t) = 0$ implies that $\mathbf{1} \perp u_t$ for all t. The space $\check{H}^u$ is covered by the ranges of the moving average processes

$$x_t = c + \sum_{k=-\infty}^{\infty} a_k u_{t-k},$$

where a_k is any square summable sequence and c any real number.

Summary. Given a white noise u_t, the space $H^u = \overline{\text{sp}}(u_t,\ t \in \mathbb{Z})$ is isomorphic with $l^2(-\infty, \infty)$. H^u is partitioned into subsets that are the ranges of moving averages of u_t. The autocovariance function γ_s of a moving average of a white noise tends to zero as $s \to \infty$.