1.4 Complex random variables and processes

Analysis of a stochastic difference equation as simple as

$$x_t = ax_{t-1} + bx_{t-2} + u_t$$

can be conveniently carried over by introducing complex stochastic variables and processes.

Basic notions and definitions regarding the field of complex numbers $\mathbb{C}$ are assumed to be known. Let us just recall that given the complex number $z = a + ib$,

$\overline{z} = a - ib$ is called the conjugate of z,
$z + \overline{z}$ is the real $2a$,
$z - \overline{z}$ is the purely imaginary $2ib$,
$z\overline{z} = a^2 + b^2 = |z|^2$, the square modulus of z (not to be confused with $z^2 = (a^2 - b^2) + 2iab$).

Moreover, there exists one and only one angle $\theta \in (-\pi\ \pi]$ (or $(0\ 2\pi]$ if you prefer) such that $a = |z| \cos\theta$ and $b = |z| \sin\theta$. This angle is referred to as the argument of z and denoted by θ_z. We have

$$z = a + ib = |z|(cos\theta_z + i \sin\theta_z).$$

The latter is called the polar representation of z, in that the point $(a,\ b)$ of the Cartesian plane is represented by means of the polar coordinates $|z|$ and θ_z. A well known trigonometric formula gives

$$zw = |z|\,|w|(\cos(\theta_z + \theta_w) + i \sin(\theta_z + \theta_w)), \tag{1.25}$$

and therefore

$$z^n = |z|^n(\cos n\theta_z + i \sin n\theta_z).$$

Thus, if θ_z does not coincide with 0 or π, the trajectory of z^t for $t \in \mathbb{N}$ describes a closing spiral, an opening spiral or remains on the unit circle according to whether the modulus of z is smaller, larger or equal to unity.

Extension of the power function, polynomial functions, or rational functions (ratios of polynomials) to the complex field are trivial. Less trivial but not difficult is the extension of other analytic functions, that is, functions that possess a Taylor expansion. In particular, we are interested in the exponential function $x \to e^x$. It is well known that the Taylor expansion of the exponential function at 0 is

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots = \sum_{j=0}^{\infty} \frac{1}{j!}x^j, \tag{1.26}$$

the series converging absolutely for all $x \in \mathbb{R}$. This implies that the series

$$1 + \frac{1}{1!}z + \frac{1}{2!}z^2 + \cdots = \sum_{j=0}^{\infty} \frac{1}{j!}z^j$$

converges absolutely, and therefore converges, for all $z \in \mathbb{C}$, so that we may give the following definition

$$e^z = 1 + \frac{1}{1!}z + \frac{1}{2!}z^2 + \cdots = \sum_{j=0}^{\infty} \frac{1}{j!}z^j,$$

which makes sense for all $z \in \mathbb{C}$.

Exercise 1.19 Be sure that you remember that a series of real or complex numbers $\sum_{k=0}^{\infty} b_n$ converges if and only if for every $\epsilon > 0$ there exists n_ϵ such that if $n > n_\epsilon$ then

$$|b_n + b_{n+1} + \cdots + b_{n+p}| < \epsilon$$

for all p (this is the Cauchy condition for series); and that if the series converges absolutely, i.e. $\sum_{k=0}^{\infty} |b_k|$ converges, then it converges.

Now observe that the fundamental identity

$$e^{x+y} = e^x\, e^y, \tag{1.27}$$

holding for all real numbers x and y, can be obtained "algebraically" from (1.26). For, given the series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$, we can consider the double sequence $a_k b_h$. If both series converge absolutely to the limits A and B respectively, then any series obtained by arbitrarily grouping the terms $a_k b_h$ (all of them), converges absolutely to AB (see [3, p. 204]). In our case, multiplication and "polynomial" grouping gives the series whose terms are

$$1$$

$$\frac{1}{1!}x + \frac{1}{1!}y = \frac{1}{1!}(x+y)$$

$$\frac{1}{2!}x^2 + \frac{1}{1!}xy + \frac{1}{2!}y^2 = \frac{1}{2!}(x+y)^2$$

$$\ldots$$

so that (1.27) is obtained in the limit. This argument can obviously be applied to the extension of the exponential to the complex field, that is, (1.27) holds for x and y belonging to $\mathbb{C}$. As a consequence,

$$e^z = e^{a+ib} = e^a\, e^{ib}.$$

On the other hand,

$$\begin{aligned} e^{ib} &= 1 + \frac{1}{1!} ib + \frac{1}{2!}(ib)^2 + \cdots \\ &= \left(1 - \frac{1}{2!}b^2 + \frac{1}{4!}b^4 + \cdots\right) + i\left(\frac{1}{1!}b - \frac{1}{3!}b^3 + \frac{1}{5!}b^5 + \cdots\right). \end{aligned}$$

Now observe that the series between brackets are the Taylor expansions of the cosine and sine function respectively, so that, in conclusion,

$$e^{a+ib} = e^a\,(\cos b + i \sin b).$$

Note that the polar representation can be rewritten as

$$z = |z| e^{i\theta_z},$$

that the rule (1.25) is easily reobtained, *without resorting to any trigonometric formula,* by observing that

$$zw = |z|\,|w|\, e^{i\theta_z} e^{i\theta_w} = |z|\,|w|\, e^{i(\theta_z+\theta_w)},$$

and that

$$z^n = |z|^n e^{in\theta_z}.$$

Exercise 1.20 Using $e^{ib} = \cos b + i \sin b$, derive

$$\cos b = \frac{e^{ib} + e^{-ib}}{2}, \qquad \sin b = \frac{e^{ib} - e^{-ib}}{2i}. \tag{1.28}$$

Using (1.29) derive the formulas for $\cos(a+b)$ and $\sin(a+b)$.

The complex exponential function maps $\mathbb{C}$ into $\mathbb{C}$. We will also make use of the function $t \to e^{i\phi t}$, mapping, for a given real ϕ, $\mathbb{Z}$, or $\mathbb{R}$, into $\mathbb{C}$ (actually the range of this function is a subset of the unit circle), and of the function $\theta \to e^{ik\theta}$, mapping, for a given integer k, the interval $[-\pi\ \pi]$, or $\mathbb{R}$, into $\mathbb{C}$.

Observation 1.15 (Periodic functions, period and frequency.) Let $f : \mathbb{R} \to \mathbb{C}$.
1. f is periodic of period $P > 0$ if $f(x + nP) = f(x)$ for all $n \in \mathbb{Z}$. Of course if P is a period for f then nP, n being a positive integer, is a period.

2. Suppose that P and Q are periods of f and $Q > P$. Then,

$$f(x + n(Q - P)) = f((x - nP) + nQ) = f(x - nP) = f(x).$$

Thus $Q - P$ is a period of f.
3. Suppose that there exists a minimum period P of f. Then if Q is a period of f, $Q = nP$, with n a positive integer. For, if $mP < Q < (m + 1)P$, for a positive integer m, then $Q - mP$ would be a period. But $Q - mP < P$. For a function without a minimum period, consider $g(x) = 1$ if x is rational, 0 if x is irrational.
4. Suppose that f has period P. The function f is periodic on $\mathbb{Z}$ if and only if one of its periods is integer, i.e. if $qP = p \in \mathbb{Z}$ for integers p and q, i.e. if $P = p/q$.
5. Let f be of period P. The number $\psi = 1/P$ is the number of cycles between x and $x + 1$, 'per unit of time' if x is interpreted as time. ϕ is called the frequency of the periodic function f.
6. The function e^{it} has minimum period 2π (the period of the cosine and sine functions) and frequency $1/2\pi$. Given any real ϕ, the period of $e^{i\phi t}$ is $2\pi/\phi$ and the frequency $\phi/2\pi$. The number ϕ is known as the *angular frequency.* It represents the number (not an integer in general) of cycles (revolutions around the unit circle) made by the complex number $e^{i\phi t}$ while t goes from 0 to 2π. The revolutions are clockwise if $\phi < 0$, anticlockwise if $\phi > 0$; if $\phi = 0$ the complex number $e^{i\phi t}$ does not move from 1. As no confusion will arise we shall refer for brevity to ϕ as "frequency" instead of "angular frequency", though the first term should be reserved to $\phi/2\pi$, the number of cycles per unit of time.
7. We only deal with discrete-time. Therefore the time-function $e^{i\phi t}$, for a real ϕ, maps $\mathbb{Z}$ into $\mathbb{C}$. As a consequence the range of ϕ can be limited to the interval $(-\pi\ \pi]$ (or any other half closed interval of the same length). For, given ϕ we have $\phi = \psi + 2k\pi$, with k integer and $\psi \in (-\pi\ \pi]$ (fairly obvious), so that

$$e^{i\phi t} = e^{i\psi t + i2\pi t} = e^{i\psi t} e^{i2k\pi t} = e^{i\psi t},$$

as, *for t integer*, $e^{i2k\pi t} = 1$. (Of course the same conclusion would not hold for continuous time.)

The definition of the vector space $\mathbb{C}^n$ is obvious. The inner product, norm and distance are defined as

$$x \cdot y = \sum_{k=1}^{n} x_k \overline{y}_k, \quad |x| = x \cdot x = \sqrt{\sum_{k=0}^{n} |x_k|^2}, \quad d(x, y) = |x - y|$$

(note the squared moduli underneath the square-root). The inner product fulfills (P1), (P3), (P4) of Section 1.3.7, whereas (P2) becomes

(P2′) $x \cdot y = \overline{y \cdot x}$.

Note that while $(ax) \cdot y = a(x \cdot y)$,

$$x \cdot (ay) = \overline{(ay) \cdot x} = \overline{a(y \cdot x)} = \overline{a}(x \cdot y). \tag{1.29}$$

Exercise 1.21 Check that least squares, orthogonal projection, Cauchy-Schwartz inequality, triangular inequality, convergence, Cauchy criterion, can be reformulated in $\mathbb{C}^n$ and all the results hold.

Exercise 1.22 Define $l^2(-\infty, \infty)$ on the complex field as the set of complex sequences $\{r_k\}$ such that $\sum_{k=-\infty}^{\infty} |r_k|^2 < \infty$. Define the inner product as $v \cdot w = \sum_{k=-\infty}^{\infty} v_k \overline{w}_k$. Check that (P1), (P3), (P4) and (P2′) hold and all the results of Section 1.3.8 can be obtained in the same way.

Now, given a measure space $(\Omega, \mathcal{F}, m)$, where m might be a probability measure, consider all the functions mapping Ω into $\mathbb{C}$, i.e. all complex functions whose domain is Ω. Obviously, a function $f : \Omega \to \mathbb{C}$ determines two functions mapping Ω into $\mathbb{R}$ such that $f(\omega) = f_1(\omega) + i f_2(\omega)$. A function f is measurable if f_1 and f_2 are measurable. It is integrable if f_1 and f_2 are integrable and in that case, by definition,

$$\int_\Omega f(\omega) dm(\omega) = \int_\Omega f_1(\omega) dm(\omega) + i \int_\Omega f_2(\omega) dm(\omega). \tag{1.30}$$

The space $L^2(\Omega, \mathcal{F}, m)$, *on the complex field,* is defined as the set of all functions $f : \Omega \to \mathbb{C}$ such that

$$\int_\Omega |f(\omega)|^2 dm(\omega) = \int_\Omega (f_1(\omega)^2 + f_2(\omega)^2) dm(\omega) < \infty$$

(note the square modulus in the integral). Inner product and norm are defined as

$$f \cdot g = \int_\Omega f(\omega)\overline{g(\omega)} dm(\omega), \quad \|f\| = \sqrt{\int_\Omega |f(\omega)|^2 dm(\omega)}, \tag{1.31}$$

the distance being defined as usual as $\|f - g\|$. Henceforth $L^2(\Omega, \mathcal{F}, m)$ will always denote the space of square integrable *complex* functions (so that heavy notation such as $L^2(\Omega, \mathcal{F}, m, \mathbb{C})$ or $L^2(\Omega, \mathcal{F}, m, \mathbb{R})$ will not be necessary).

Exercise 1.23 Starting with the fact that integrals are defined by means of sums, extend the definition of integral to functions taking values in $\mathbb{C}$ and show that (1.30) is a consequence of the definition. Observe that all we need to define the integral of a function $F : \Omega \to V$ is that V is a vector space on the real or the complex field, so that sum and multiplication by a scalar are defined, and that V has a metric, so that convergence is defined. For example, V might be a Hilbert space. Show that if V is a finite-dimensional vector space, so that

$$F(\omega) = (F_1(\omega),\ F_2(\omega),\ \dots,\ F_n(\omega)),$$

the integral of F is the vector whose components are the integrals of the components of F. We will come back to this point when defining stochastic integrals in Chapter 3.

Exercise 1.24 Check that (P1), (P3), (P4) and (P2$'$) hold for the space $L^2(\Omega, \mathcal{F}, m)$, on the complex field, with the inner product defined in (1.31), and that all the statements of Section 1.3.9 can be obtained in the same way.

Quite obviously, the definition of a general Hilbert space on the complex field only requires axioms (P1), (P3), (P4) and (P2$'$). Obviously, both $\ell^2(-\infty, \infty)$ and $L^2(\Omega, \mathcal{F}, m)$, on the complex field, are Hilbert spaces. All the statements in Section 1.3.10 hold. Note that (1.21) becomes

$$v \cdot w = \sum_{k=-\infty}^{\infty} (v \cdot v_k)\overline{(w \cdot v_k)}. \tag{1.32}$$

For, consider that

$$\begin{aligned} v \cdot w &= \lim_{m\to\infty} \left(\sum_{k=-m}^{m} (v \cdot v_k) v_k \right) \left(\sum_{k=-m}^{m} (w \cdot v_k) v_k \right) \\ &= \lim_{m\to\infty} \left(\sum_{k=-m}^{m} [(v \cdot v_k) v_k] \cdot [(w \cdot v_k) v_k] \right). \end{aligned}$$

The conclusion follows from (1.29). Thus the map mapping v to the sequence of its Fourier coefficients establishes an isomorphism between M and the space $l^2(-\infty, \infty)$ on the complex field. Lastly, note that (1.20) takes the form

$$\|v\|^2 = \sum_{k=-\infty}^{\infty} |v \cdot v_k|^2. \tag{1.33}$$

Now assume that the measure space is a probability space $(\Omega, \mathcal{F}, P)$. The random variable $z : \Omega \to \mathbb{C}$ determines two real random variables x and y such that $z = x + iy$. The distribution of z is defined as the distribution of the real vector $(x,\ y)$. Given a vector $Z = (z_1,\ z_2,\ \ldots,\ z_n)$ of complex random variables on $(\Omega, \mathcal{F}, P)$, with $z_j = x_j + iy_j$, the distribution of Z is the distribution of the real vector

$$(x_1,\ x_2,\ \ldots,\ x_n,\ y_1,\ y_2,\ \ldots,\ y_n).$$

The variables z_1 and z_2 are independent if

$$P\left[(x_1, y_1, x_2, y_2) \in A \times B\right] = P\left[(x_1, y_1) \in A)\right] P\left[(x_2, y_2) \in B)\right],$$

where A and B are subset of $\mathbb{R}^2$.

Exercise 1.25 Show that if $z = z_1 + iz_2$ and $w = w_1 + iw_2$ are independent then z_j and w_k are independent for all $j, k = 1, 2$ (thus independence of z and w implies that $\operatorname{cov}(z_j, w_k) = 0$ for all $j, k = 1, 2$). The converse is not true, i.e. independence of z_j and w_k for all $j, k = 1, 2$ does not imply that z and w are independent. For, consider the following stochastic variables, where $\Omega = [0\ 1]$ with the uniform distribution:

$$a_1(\omega) = \begin{cases} 1 \text{ for } \omega < \frac{1}{2} \\ 0 \text{ otherwise} \end{cases} \qquad a_2(\omega) = \begin{cases} 1 \text{ if } \omega < \frac{1}{4} \\ 1 \text{ if } \frac{1}{2} \le \omega < \frac{3}{4} \\ 0 \text{ otherwise} \end{cases} \qquad a_3(\omega) = \begin{cases} 1 \text{ if } \omega < \frac{1}{4} \\ 1 \text{ if } \omega \ge \frac{3}{4} \\ 0 \text{ otherwise} \end{cases}$$

Prove that a_j and a_k are independent where $j \neq k,\ j, k = 1, 3$. However, a_1 and the vector $(a_2\ a_3)$ are not independent. Indeed

$$P(a_1 = 1 \text{ and } (a_2\ a_3) = (1\ 1)) = P(a_1 = 1,\ a_2 = 1,\ a_3 = 1) = 1/4,$$

whereas

$$P(a_1 = 1)\ P((a_2\ a_3) = (1\ 1)) = P(a_1 = 1)\ P(a_2 = 1)\ P(a_3 = 1) = 1/8$$

(this example is adapted from [9], p. 195, exercise (4)). Thus if we define $z = a_1 + ia_1$ and $w = a_2 + ia_3$, z and w are not independent although z_j and w_k are independent for all $j, k = 1, 2$.

The space $L^2(\Omega, \mathcal{F}, P)$ over the complex field is the obvious specification of $L^2(\Omega, \mathcal{F}, m)$ over the complex field. Note that the definition of covariance for complex random variables is

$$\operatorname{cov}(x, y) = E\left[(x - E(x)\overline{(y - E(y))}\right] = \int_{\Omega} (x(\omega) - E(x)\overline{(y(\omega) - E(y)}dP(\omega),$$

so that $\operatorname{cov}(x, y) = \overline{\operatorname{cov}(y, x)}$ and

$$\operatorname{var}(x) = \operatorname{cov}(x, x) = \int_{\Omega} |x(\omega) - E(x)|^2 dP(\omega).$$

Of course, convergence of x_n to x in mean square implies convergence to zero of both the variance and the mean of $x - x_n$ (see Observation 1.7).

The definitions of complex stochastic process, stationary and weak stationary complex processes do not present any difficulty. Note that for complex weakly stationary processes we have

$$\gamma_k = \operatorname{cov}(x_t, x_{t-k}) = \overline{\operatorname{cov}(x_{t-k}, x_t)} = \overline{\operatorname{cov}(x_t, x_{t+k})} = \overline{\gamma}_{-k}.$$

Example 1.12 (Complex version of Example 1.3) Let A be a zero-mean complex stochastic variable belonging to $L^2(\Omega, \mathcal{F}, P)$ and ϕ a real number. Define

$$x_t = Ae^{i\phi t}. \tag{1.34}$$

It is easy to prove that x_t is a weakly stationary process:

$$E(x_t) = E(A)e^{i\phi t} = 0, \quad \gamma_k = E\left(Ae^{i\phi t}\overline{Ae^{i\phi(t-k)}}\right) = \sigma_A^2 e^{i\phi k}.$$

Exercise 1.26 Let A_j, $j = 1, 2, \ldots, n$, be zero-mean mutually orthogonal stochastic variables belonging to $L^2(\Omega, \mathcal{F}, P)$, i.e. $E(A_j \overline{A}_k) = 0$. Let ϕ_j, $j = 1, 2, \ldots, n$, be real numbers and define

$$x_t = A_1 e^{i\phi_1 t} + A_2 e^{i\phi_2 t} + \cdots + A_n e^{i\phi_n t}. \tag{1.35}$$

Prove that x_t is weakly stationary and determine its autocovariance function.

Exercise 1.27 In Exercise 1.26 assume $n = 2$, $A_2 = \overline{A}_1$, $\phi_2 = -\phi_1$. Setting $A_1 = a + ib$, prove that the orthogonality assumption on $A_1 \perp A_2$, i.e. $A_1 \perp \overline{A}_1$, implies that $\operatorname{var}(a) = \operatorname{var}(b)$, $\operatorname{cov}(a, b) = 0$, and that

$$x_t = 2(a \cos\phi t - b \sin\phi t),$$

which is Example 1.3, up to a trivial redefinition.

Exercise 1.28 Assume that A and ϕ are independent stochastic variables on the probability space $(\Omega, \mathcal{F}, P)$, that A has zero mean, ϕ is real and define

$$x_t = Ae^{i\phi t}.$$

Prove that if A has finite second moment then x_t is weakly stationary. Obtain Example 1.4 in the same way as Example 1.3 is obtained in Example 1.27. (For this example see [7], p. 479, Example 4.)

Exercise 1.29 Given two complex stochastic variables z and w on $(\Omega, \mathcal{F}, P)$, let

$$z = z_1 + i z_2, \quad w = w_1 + i w_2,$$

where z_j and w_j, $j = 1, 2$, are real stochastic variables on $(\Omega, \mathcal{F}, P)$. Show that if $\mathrm{cov}(z_j, w_k) = 0$, for $j, k = 1, 2$, then $\mathrm{cov}(z, w) = 0$, but that the converse statement is not true. An example is $w = \bar{z}$, under the conditions $\mathrm{cov}(z_1, z_2) = 0$, $\mathrm{var}(z_1) = \mathrm{var}(z_2)$. In this case zero-correlation of z and w does not imply $\mathrm{cov}(z_j, w_k) = 0$ for all $j, k = 1, 2$ (nor independence of z and w; see Exercise 1.25).

Summary. We have briefly reviewed basic notions on complex numbers and introduced the complex function e^z by extending the Taylor expansion of the real exponential function. Complex vector spaces, Hilbert spaces, the complex function space L^2, complex random variables and processes have been defined. The basic results on convergence and completeness can be proved by fairly obvious modifications of their real counterparts.